

Determination of the $\eta^3\text{He}$ threshold structure from the low energy $pd \rightarrow \eta^3\text{He}$ reactionJu-Jun Xie,^{1,2,*} Wei-Hong Liang,^{3,†} Eulogio Oset,^{1,2} Pawel Moskal,⁴ Magdalena Skurzok,⁴ and Colin Wilkin⁵¹*Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China*²*Departamento de Física Teórica and IFIC, Centro Mixto Universidad de Valencia–CSIC,**Institutos de Investigación de Paterna, Apartado 22085, 46071 Valencia, Spain*³*Department of Physics, Guangxi Normal University, Guilin 541004, China*⁴*Institute of Physics, Jagiellonian University, ul. Stanistawa Łojasiewicza 11, 30-348 Kraków, Poland*⁵*Department of Physics and Astronomy, University College London, London WC1E 6BT, United Kingdom*

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We analyze the data on cross sections and asymmetries for the $pd \rightarrow \eta^3\text{He}$ reaction close to threshold and look for bound states of the $\eta^3\text{He}$ system. Rather than parameterizing the scattering matrix, as is usually done, we develop a framework in which the $\eta^3\text{He}$ optical potential is the key ingredient, and its strength, together with some production parameters, are fitted to the available experimental data. The relationship of the scattering matrix to the optical potential is established using the Bethe-Salpeter equation and the $\eta^3\text{He}$ loop function incorporates the range of the interaction given by the empirical ^3He density. We find a local Breit-Wigner form of the $\eta^3\text{He}$ amplitude T below threshold with a clear peak in $|T|^2$, which corresponds to an $\eta^3\text{He}$ binding of about 0.3 MeV and a width of about 3 MeV. By fitting the potential we can also evaluate the $\eta^3\text{He}$ scattering length, including its sign, thus resolving the ambiguity in the former analyses.

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The identification of η bound states in nuclei has been a constant effort for several years [1–6], starting from the early works of Refs. [7–9]. More precise evaluations of the η -nucleus optical potential, with special attention to two-nucleon η absorption, indicated that, while indeed the ηN interaction was strong enough to bind η states, the widths were always bigger than the binding [10].

An important step forward was made possible by the advent of chiral unitary theory to describe the meson-baryon interaction [11–15]. Within this theoretical framework, the ηN interaction with coupled channels was studied in Refs. [11,16], and the $N^*(1535)$ resonance appeared dynamically generated in the scheme, albeit with unnatural subtraction constants in the regularization of the loops. This deficiency is an indication that other components are missing in the approach [17,18], and the problem was solved in Ref. [19] through the inclusion of the ρN and $\pi\Delta$ channels, in addition to the $\pi N, \eta N, K\Lambda, K\Sigma$ of the original chiral unitary approach.

The η -nucleus interaction within the chiral unitary approach was studied in Ref. [20], where enough attraction was found to form bound η -nucleus states. Detailed studies of the η energies for different nuclei were made in Ref. [21] where, for medium and light nuclei, bound states were found (see also Ref. [22], where qualitatively similar conclusions were drawn), though with larger widths than binding energies. For instance, for ^{12}C , binding $B_E = 9.7$ MeV and width $\Gamma = 35$ MeV were found with the preferred energy-dependent potential. Assuming that these two magnitudes scale roughly with the mass number, for ^3He we could expect $B_E \simeq 2.4$ MeV, $\Gamma \simeq 8.7$ MeV, though

the width could be somewhat smaller than this because the relative weight of two-body η absorption should be smaller in ^3He than in ^{12}C . On the other hand, some theoretical calculations for light systems predict B_E of around 1 MeV or less and $\Gamma = 15$ MeV for $\eta^3\text{He}$ [23]. The fact that the widths are expected to be much larger than the binding might be the reason why, so far, we have no conclusive evidence for any of these bound states [24–38].

The data on the $pd(dp) \rightarrow \eta^3\text{He}$ total cross section show a sharp rise from threshold before becoming stable at an excess energy of about $Q = 1$ MeV, keeping this constant value up to about 10 MeV [25,39]. These data have been analyzed before in Refs. [25,26]. In Ref. [25] only an s -wave amplitude for $\eta^3\text{He}$ was considered, while in Ref. [26] the s -wave and p -wave interference data were considered in order to further constrain the $\eta^3\text{He}$ amplitude. This analysis suggested a pole with a binding energy of around 0.3 MeV and with a very small width.

In the present work, we describe an alternative method of analysis, following the algorithms used in the chiral unitary approach. This allows one to produce an $\eta^3\text{He}$ amplitude that is fully unitary and with proper analytical properties, without the approximations, or assumptions, made in Refs. [25,26]. Our approach does not presume any particular form of the amplitude but rather generates it from an $\eta^3\text{He}$ potential, which is what is fitted to the data. The T matrix then arises from the solution of the Lippmann-Schwinger equation, though, for convenience, we use the Bethe-Salpeter equation (BSE), which allows us to keep relativistic terms.

In our approach we can relate the parameters of the potential to the ηN scattering length and this provides a valuable constraint. The ηN scattering length in the chiral unitary approach is estimated to be $a_{\eta N} = (-0.264 - i0.245)$ fm in Ref. [20] and $a_{\eta N} = (-0.20 - i0.26)$ fm in Ref. [11]. Other approaches include the results of Ref. [40] with

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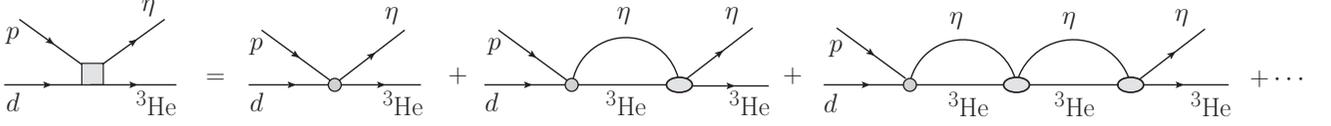


FIG. 1. The process $pd \rightarrow \eta^3\text{He}$ considering explicitly the $\eta^3\text{He}$ rescattering. The square box in the first diagram indicates the full transition amplitude, while the circle in the second diagram stands for the bare transition amplitude prior to the $\eta^3\text{He}$ final state interaction. It contains all diagrams that do not have $\eta^3\text{He}$ as intermediate state. The oval stands for the $\eta^3\text{He}$ optical potential.

$a_{\eta N} = (-0.87 - i0.27)$ fm and those of Ref. [41] with $a_{\eta N} = (-0.691 - i0.174)$ fm in one option and $a_{\eta N} = (-0.968 - i0.281)$ fm in another. A different version by the same group yielded $a_{\eta N} = (-0.910 \pm 0.050 - i(0.290 \pm 0.04))$ fm [42]. An interesting result is the constraint on $\text{Im}(a_{\eta N})$ from the optical theorem and the inelastic cross section of $\pi N \rightarrow \eta N$ [42], $|\text{Im}(a_{\eta N})| \geq (0.24 \pm 0.02)$ fm.

Some parameters that we fit to the data can be related, at least approximately, to $a_{\eta N}$, and this will be used as a consistency check of the results. As we shall see later, the output of our calculations leads to an $\eta^3\text{He}$ optical potential from which we deduce a value of $a_{\eta N}$ that is basically consistent with experimental information. With this optical potential we solved the BSE for the $\eta^3\text{He}$ system and found an $\eta^3\text{He}$ bound state with a binding energy of around 0.3 MeV and a width of around 3 MeV, with reasonable uncertainties. We can therefore claim that the data of the $pd(dp) \rightarrow \eta^3\text{He}$ reaction close to threshold provide evidence of a very weakly bound $\eta^3\text{He}$ state.

Steps in a similar direction to ours were taken in Ref. [30], where the available data on the $\eta^3\text{He}$ and $\eta^4\text{He}$ systems were simultaneously fit in terms of optical potentials. There are, however, significant differences in the approaches as well as in the quality of the present $pd \rightarrow \eta^3\text{He}$ data, and these are discussed in Sec. V.

II. FORMALISM

The $\eta^3\text{He}$ interaction

Let us depict diagrammatically the $pd \rightarrow \eta^3\text{He}$ process. This is done in Fig. 1.

The $\eta^3\text{He}$ scattering amplitude is given by the diagrams depicted in Fig. 2, and formally by

$$T = V + VGT, \quad (1)$$

where V is the $\eta^3\text{He}$ optical potential, which contains an imaginary part to account for the inelastic channels $\eta^3\text{He} \rightarrow X$, where X is mostly $\pi 3N$. It also includes the $3N$ intermediate state arising mainly from η two-body absorption, $\eta NN \rightarrow NN$ [10,20,43].

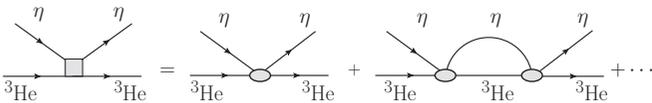


FIG. 2. Diagrammatic representation of the $\eta^3\text{He}$ scattering matrix.

In many-body theory it is known that at low densities the optical potential is given by

$$V(\vec{r}) = 3t_{\eta N}\tilde{\rho}(\vec{r}), \quad (2)$$

where $t_{\eta N}$ is the forward ηN amplitude and $\tilde{\rho}(\vec{r})$ is the ^3He density normalized to unity. Equation (2) is relatively accurate in many-body physics, but here we are more concerned with the fact that it provides the realistic range of η -nucleus interaction, since the η can interact with all the nucleons in the nucleus distributed according to $\rho(\vec{r})$.

In momentum space the potential is given by

$$\begin{aligned} V(\vec{p}_\eta, \vec{p}'_\eta) &= 3t_{\eta N} \int d^3\vec{r} \tilde{\rho}(\vec{r}) e^{i(\vec{p}_\eta - \vec{p}'_\eta) \cdot \vec{r}} \\ &= 3t_{\eta N} F(\vec{p}_\eta - \vec{p}'_\eta), \end{aligned} \quad (3)$$

where $F(\vec{q})$ is the ^3He form factor,

$$F(\vec{q}) = \int d^3\vec{r} \tilde{\rho}(\vec{r}) e^{i\vec{q} \cdot \vec{r}}, \quad (4)$$

and $F(\vec{0}) = 1$. A good approximation to this form factor at small momentum transfers is given by a Gaussian,

$$F(\vec{q}) = e^{-\beta^2 |\vec{q}|^2}, \quad (5)$$

where $\beta^2 = \langle r^2 \rangle / 6$. This mean-square radius corresponds to the distribution of the centers of the nucleons and, after correcting for the nucleon size, this leads to an experimental value of $\beta^2 = 13.7 \text{ GeV}^{-2}$ [44].

Due to the form factor, the optical potential in Eq. (3) contains all partial waves. There are other sources for the p waves and these will be treated later in an empirical way. After integrating over the angle between \vec{p}'_η and \vec{p}_η , the s wave projection of the optical potential becomes

$$\begin{aligned} V(\vec{p}_\eta, \vec{p}'_\eta) &= 3t_{\eta N} \frac{1}{2} \int_{-1}^1 d \cos \theta e^{-\beta^2(|\vec{p}_\eta|^2 + |\vec{p}'_\eta|^2 - 2|\vec{p}_\eta||\vec{p}'_\eta| \cos \theta)} \\ &= 3t_{\eta N} e^{-\beta^2|\vec{p}_\eta|^2} e^{-\beta^2|\vec{p}'_\eta|^2} \\ &\quad \times \left[1 + \frac{1}{6} (2\beta^2|\vec{p}_\eta||\vec{p}'_\eta|)^2 + \dots \right]. \end{aligned} \quad (6)$$

The term $2\beta^2|\vec{p}_\eta||\vec{p}'_\eta|/6$ is negligible in the region where $e^{-\beta^2|\vec{p}_\eta|^2} e^{-\beta^2|\vec{p}'_\eta|^2}$ is sizable and can be neglected, and this leads to a potential that is separable in the variables \vec{p}_η and \vec{p}'_η , which makes the solution of Eq. (1) trivial. Keeping the relativistic

factors of the Bethe-Salpeter equation, we can write [12]

$$T(\vec{p}_\eta, \vec{p}'_\eta) = \tilde{V} e^{-\beta^2 |\vec{p}_\eta|^2} e^{-\beta^2 |\vec{p}'_\eta|^2} + \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{M_{3\text{He}}}{2\omega_\eta(\vec{q}) E_{3\text{He}}(\vec{q})} \tilde{V} e^{-\beta^2 |\vec{p}_\eta|^2} e^{-\beta^2 |\vec{q}|^2} \times \frac{1}{\sqrt{s} - \omega_\eta(\vec{q}) - E_{3\text{He}}(\vec{q}) + i\epsilon} T(\vec{q}, \vec{p}'_\eta), \quad (7)$$

with \sqrt{s} being the invariant mass of the $\eta^3\text{He}$ system, $\omega_\eta(\vec{q}) = \sqrt{m_\eta^2 + |\vec{q}|^2}$, and $E_{3\text{He}}(\vec{q}) = \sqrt{M_{3\text{He}}^2 + |\vec{q}|^2}$. We have here taken \tilde{V} instead of $3t_{\eta N}$ for more generality.

By expanding T in Eq. (7) as $V + VGV + VGVGV + \dots$ we can see that all terms with G contain $e^{-2\beta^2 |\vec{q}|^2}$ and that the factors $e^{-\beta^2 |\vec{p}_\eta|^2} e^{-\beta^2 |\vec{p}'_\eta|^2}$ can be factorized outside the integral. Hence, the T matrix can be factorized in the same way as V , and we have

$$T(\vec{p}_\eta, \vec{p}'_\eta) = \tilde{T} e^{-\beta^2 |\vec{p}_\eta|^2} e^{-\beta^2 |\vec{p}'_\eta|^2}. \quad (8)$$

The Bethe Salpeter equation becomes then algebraic,

$$\tilde{T} = \tilde{V} + \tilde{V}G\tilde{T}, \quad (9)$$

with

$$G = \frac{M_{3\text{He}}}{16\pi^3} \int \frac{d^3\vec{q}}{\omega_\eta(\vec{q}) E_{3\text{He}}(\vec{q})} \frac{e^{-2\beta^2 |\vec{q}|^2}}{\sqrt{s} - \omega_\eta(\vec{q}) - E_{3\text{He}}(\vec{q}) + i\epsilon}. \quad (10)$$

In Fig. 3, we show the real and imaginary parts of the loop function G as a function of the excess energy Q ($Q = \sqrt{s} - m_\eta - M_{3\text{He}}$).

In the normalization that we are using, the η -nucleon and $\eta^3\text{He}$ scattering lengths are related to the t matrices by

$$a_{\eta N} = \frac{1}{4\pi} \frac{m_N}{\sqrt{s_{\eta N}}} t_{\eta N} \Big|_{\sqrt{s_{\eta N}}=m_N+m_\eta} \quad (11)$$

$$a_{\eta^3\text{He}} = \frac{1}{4\pi} \frac{M_{3\text{He}}}{\sqrt{s}} T \Big|_{\sqrt{s}=M_{3\text{He}}+m_\eta}. \quad (12)$$

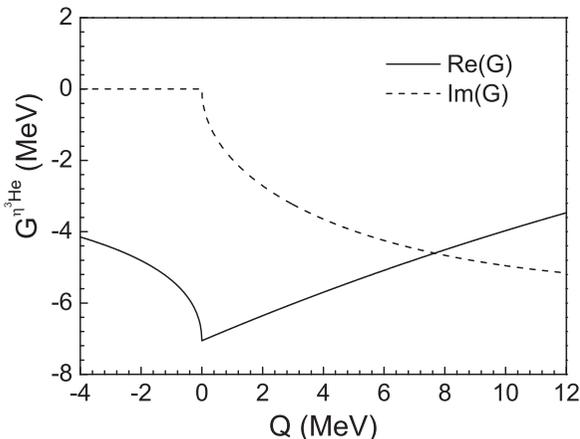


FIG. 3. Real (solid line) and imaginary (dashed line) parts of the G of Eq. (10) as functions of the excess energy Q .

The strategy that we adopt is to fit \tilde{V} to the $pd(dp) \rightarrow \eta^3\text{He}$ data and then see how different \tilde{V} is from $3t_{\eta N}$ by evaluating

$$a'_{\eta N} = \frac{1}{4\pi} \frac{m_N}{\sqrt{s_{\eta N}}} \frac{\tilde{V}}{3} \Big|_{\sqrt{s_{\eta N}}=m_N+m_\eta} \quad (13)$$

and comparing it to the theoretical value of $a_{\eta N}$.

After obtaining the *best* value for \tilde{V} , we then plot

$$T = \tilde{T} e^{-2\beta^2 |\vec{p}_\eta|^2} \quad (14)$$

and investigate $|T|^2$ below threshold to identify a bump and its width. From this we can determine roughly the position and width of the bound state. The more precise determination is done by plotting $\text{Re}(T)$ and $\text{Im}(T)$, and we see that in a narrow range of Q they are consistent with a Breit-Wigner form

$$T = \frac{g^2}{\sqrt{s} - M_R + i\Gamma/2} = \frac{g^2(\sqrt{s} - M_R)}{(\sqrt{s} - M_R)^2 + \Gamma^2/4} - i \frac{g^2\Gamma/2}{(\sqrt{s} - M_R)^2 + \Gamma^2/4}, \quad (15)$$

where g is constant and M_R and Γ are the mass and width of the $\eta^3\text{He}$ bound state.

III. PRODUCTION AMPLITUDE IN THE s WAVE

Following the formalism of Ref. [26], we write for the $pd \rightarrow \eta^3\text{He}$ transition depicted as a circle in Fig. 1

$$V_P = A\vec{\epsilon} \cdot \vec{p} + iB(\vec{\epsilon} \times \vec{\sigma}) \cdot \vec{p}, \quad (16)$$

where $\vec{\epsilon}$ is the polarization of the deuteron, $\vec{\sigma}$ denotes the Pauli matrix standing for the spin of the proton, and \vec{p} is the momentum in the initial state. This amplitude has the initial-state p wave needed to match the $\eta(0^-)^3\text{He}(1/2^+)$ with the $d(1^+)p(1/2^+)$ system. In Ref. [26] $A = B$ was taken, which is consistent with the SPESIV experiment [45], but there is no loss of generality for the total cross section if another choice is made.

Some extra information on these parameters is obtained from Ref. [46], where the ratio $|A|/|B|$ was found to be constant, of the order of 0.9 for $Q < 10$ MeV. The choice $A = B$ can be interpreted as having the dp system in spin 1/2, according to the study done in Ref. [47], where the analogous conjugate reaction $\Lambda_b \rightarrow J/\psi K^0 \Lambda$ was studied.

With similar arguments to those used to derive Eq. (3), we can justify that V_P in Eq. (16) must be accompanied by the factor $e^{-\beta^2 |\vec{p}_\eta|^2}$, which, if the η is in the loop, will become $e^{-\beta^2 |\vec{q}|^2}$. In view of this we can write analytically the equation for the diagrams of Fig. 1 as

$$t_{pd \rightarrow \eta^3\text{He}} = V_P e^{-\beta^2 |\vec{p}_\eta|^2} + V_P G \tilde{T} e^{-\beta^2 |\vec{p}_\eta|^2} = V_P e^{-\beta^2 |\vec{p}_\eta|^2} (1 + G\tilde{T}) = \frac{V_P e^{-\beta^2 |\vec{p}_\eta|^2}}{1 - \tilde{V}G}, \quad (17)$$

where in the last step we have used Eq. (1). The cross section then becomes

$$\sigma = \frac{m_p M_{3\text{He}}}{12\pi s} (|A|^2 + 2|B|^2) |\vec{p}_\eta| |\vec{p}| e^{-2\beta^2 |\vec{p}_\eta|^2}, \quad (18)$$

with

$$A' = \frac{A}{1 - \tilde{V}G}, \quad B' = \frac{B}{1 - \tilde{V}G}. \quad (19)$$

This means that, in the analysis of the total cross section with only s waves, we can, without loss of generality, take $A = B$ and real. This will allow us to perform a fit to the data up to an excess energy $Q = 2$ MeV, where the p -wave effects are negligible, and thus determine \tilde{V} . From these, by means of Eq. (9), we shall determine T and investigate its structure below threshold.

IV. THE INCLUSION OF p WAVES

Following once again the approach of Ref. [26], we assume an η - ^3He p -wave production amplitude

$$V_{1P} = C\vec{\epsilon} \cdot \vec{p}_\eta + iD(\vec{\epsilon} \times \vec{\sigma}) \cdot \vec{p}_\eta. \quad (20)$$

This amplitude will be taken empirically and once again there is support for $C = D$ from the experiment of Ref. [45]. Hence we take $A = B$ and $C = D$ as in Ref. [26].

As in Ref. [26] we shall also take into account the s - and p -wave interference by means of the asymmetry parameter α defined as

$$\alpha = \frac{d}{d \cos \theta_\eta} \ln \left(\frac{d\sigma}{d\Omega} \right) \Big|_{\cos \theta_\eta = 0}. \quad (21)$$

By means of Eqs. (17) and (20) we obtain

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{m_p M_{^3\text{He}}}{48\pi^2 s} \frac{|\vec{p}_\eta|}{|\vec{p}|} ((|A'|^2 + 2|B'|^2) |\vec{p}|^2 e^{-\beta^2 |\vec{p}_\eta|^2} \\ &+ (|C|^2 + 2|D|^2) |\vec{p}_\eta|^2 + 2 \text{Re}(A'C^* + 2B'D^*) \\ &\times |\vec{p}| |\vec{p}_\eta| \cos(\theta_\eta)), \end{aligned} \quad (22)$$

from which we find that

$$\alpha = \frac{2 \text{Re}(A'C^* + 2B'D^*) |\vec{p}| |\vec{p}_\eta|}{(|A'|^2 + 2|B'|^2) |\vec{p}|^2 e^{-2\beta^2 |\vec{p}_\eta|^2} + (|C|^2 + 2|D|^2) |\vec{p}_\eta|^2}. \quad (23)$$

In addition, the total cross section of Eq. (18) becomes

$$\begin{aligned} \sigma &= \frac{m_p M_{^3\text{He}}}{12\pi s} \frac{|\vec{p}_\eta|}{|\vec{p}|} ((|A'|^2 + 2|B'|^2) |\vec{p}|^2 e^{-2\beta^2 |\vec{p}_\eta|^2} (|C|^2 \\ &+ 2|D|^2) |\vec{p}_\eta|^2). \end{aligned} \quad (24)$$

Equations (23) and (24) are used to fit the experimental data on the $dp \rightarrow \eta^3\text{He}$ total cross section and asymmetry parameter α of Ref. [25].

The value of \tilde{V} obtained from the s -wave analysis was used as a starting value for the global fit, but the resulting parameter does not differ significantly from that found in the s -wave analysis.

V. RESULTS

A. s -wave analysis

First, we perform three-parameter [$A = B = r_A$ and $\tilde{V} = \text{Re}(V) + i \text{Im}(V)$] χ^2 fits to the experimental data on the total cross sections of the $pd(dp) \rightarrow \eta^3\text{He}$ reaction below $Q =$

TABLE I. Values of parameters determined in this work.

| Parameter | Fitted value |
|-------------------------------|------------------------------------|
| r_A (MeV $^{-2}$) | $(9.43 \pm 0.17) \times 10^{-7}$ |
| r_C (MeV $^{-2}$) | $(6.85 \pm 0.31) \times 10^{-6}$ |
| θ (degree) | 347 ± 2 |
| γ (MeV $^{-1}$) | $(-5.25 \pm 0.15) \times 10^{-2}$ |
| $\text{Re}(V)$ (MeV $^{-1}$) | $(-14.57 \pm 0.42) \times 10^{-2}$ |
| $\text{Im}(V)$ (MeV $^{-1}$) | $(-5.36 \pm 0.14) \times 10^{-2}$ |

2 MeV. The main purpose of this part of the fit is to provide starting values for the parameters of the global fit, where the full measured range in Q is used.

B. Results including the p wave

We next perform six-parameter [$A = B = r_A$, $C = D = r_C e^{i\theta}(1 + \gamma Q)$, and $\tilde{V} = \text{Re}(V) + i \text{Im}(V)$] χ^2 fits to the experimental data on the total cross sections and asymmetry of the $pd \rightarrow \eta^3\text{He}$ reaction [25]. The values of the resulting parameters are collected in Table I.

With the potential \tilde{V} obtained from these fits, we have evaluated the scattering length $a'_{\eta N}$ of Eq. (13):

$$a'_{\eta N} = [-(0.48 \pm 0.05) - i(0.18 \pm 0.02)] \text{ fm}. \quad (25)$$

These results are very close to those of the $a_{\eta N}$ scattering length determined in different studies, and fair enough for the approximation introduced in the low density theorem of Eq. (2), thus giving support to the analysis done here.

Similarly, by means of Eq. (12), we determine the $\eta^3\text{He}$ scattering length to be

$$a_{\eta^3\text{He}} = [(2.23 \pm 1.29) - i(4.89 \pm 0.57)] \text{ fm}. \quad (26)$$

Note that the strategy of fitting an optical potential to the data instead of the usual t -matrix parametrization used in previous works, allows us to determine the sign of the real part of the scattering lengths. The fit yields an attractive potential, which is consistent with all theoretical derivations of $t_{\eta N}$, together with the $t_{\eta N} \tilde{\rho}(\vec{r})$ assumption for the optical potential.

It is interesting to see that the errors for $a_{\eta^3\text{He}}$ are relatively large. This is in fact not surprising, since in Ref. [26] $a_{\eta^3\text{He}} = (\pm 10.9 - i 1.0)$ fm was obtained from the ANKE data, while in Ref. [39] the COSY-11 Collaboration reported $a_{\eta^3\text{He}} = (\pm 2.9 - i 3.2)$ fm, though without taking beam smearing effects into account. Nevertheless, the two raw data sets are compatible.

It is also interesting to note that results for $a_{\eta^3\text{He}}$ similar to those in Eq. (26), $(2.3, -i 3.2)$ fm, were found from the simultaneous analysis of much cruder data for the $pd \rightarrow \eta^3\text{He}$ and $dd \rightarrow \eta^4\text{He}$ reactions. The combined fit was deemed necessary because *the error correlation between the real and imaginary parts of $a_{\eta^3\text{He}}$ makes it hard to fix both parameters purely from the $\eta^3\text{He}$ production data* [30]. The $\eta^3\text{He}$ and $\eta^4\text{He}$ optical potentials were taken to be proportional to $A \rho(r)$, where $\rho(r)$ is the nuclear density and A the number of nucleons. However, the agreement is in large part coincidental

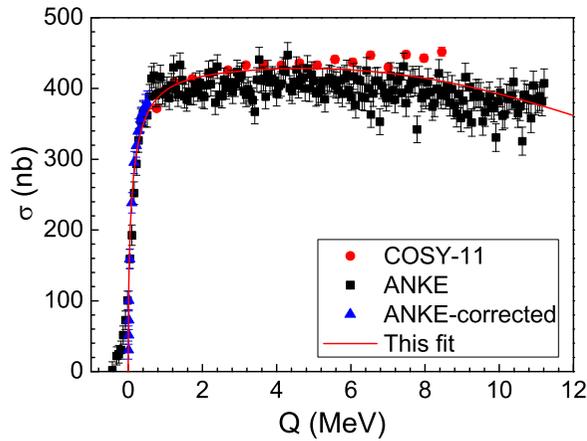


FIG. 4. The fitted $dp \rightarrow \eta^3\text{He}$ total cross sections compared with experimental data [25,39].

since it was assumed in Ref. [30] that the data could be parametrized as

$$\frac{d\sigma}{d\Omega} = \frac{|\vec{p}_\eta|}{|\vec{p}_d|} |f|^2, \quad f = \frac{f_B}{1 + ia_{\eta^3\text{He}} p_\eta}, \quad (27)$$

with f_B constant and $a_{\eta^3\text{He}}$ the $\eta^3\text{He}$ scattering length. As can be seen from Eqs. (20), (22), and (24), our prediction for the structure of the cross section is much richer than that of Eq. (27), since the $\eta^3\text{He}$ interaction includes implicitly the effective range and higher order terms, as well as relativistic corrections.

The fitted total cross sections reproduce well the experimental data shown in Fig. 4. In Fig. 5 we show the detail of this fit in the $Q < 2$ MeV region. Note that Eq. (24) is only valid for $Q > 0$. Due to experimental resolution and beam momentum spread, some data in Figs. 4 and 5 appear below threshold. These data are corrected by inverting the implicit convolution of the real cross sections with the experimental resolution as carried out by the ANKE Collaboration (see more details in Ref. [25]). The corrected data are shown in Figs. 4 and 5 by

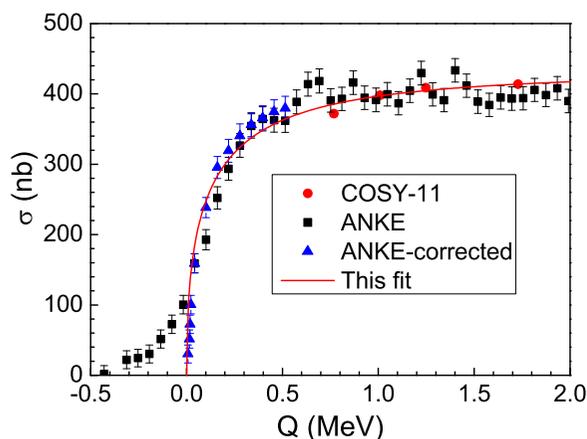


FIG. 5. The fitted $dp \rightarrow \eta^3\text{He}$ total cross sections below $Q = 2$ MeV compared with experimental data [25,39].

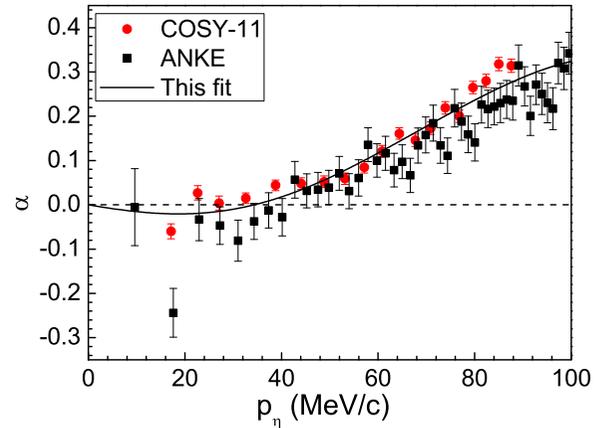


FIG. 6. The fit in the model to the asymmetry parameter α as a function of the center-of-mass η momentum p_η compared with the experimental data [25,39].

blue triangles. These are obtained by shifting the experimental data to the deconvoluted distribution of Ref. [25].

The fitted results of the asymmetry parameter α shown in Fig. 6 also describe well the experimental data.

We next turn our attention to the $\eta^3\text{He} \rightarrow \eta^3\text{He}$ scattering amplitude. In Fig. 7, we depict $|T|^2$ obtained with the fitted parameters given in Table I, as a function of Q . We see a qualitative picture of a very weakly bound state of $\eta^3\text{He}$ system.

In order to make more quantitative statements, we plot in Fig. 8 the real and imaginary parts of T . We see that at $Q = -0.3$ MeV, $\text{Re}(T)$ goes from negative to positive passing through zero, $\text{Im}(T)$ is negative, and $-\text{Im}(T)$ has a peak. In Fig. 8 we show also the results obtained with the Breit-Wigner form of Eq. (15)¹ by red-solid and red-dashed curves as a function of Q from -1 to 0 MeV. We see that in a narrow

¹The coupling g is fitted to the imaginary part of $\eta^3\text{He} \rightarrow \eta^3\text{He}$ scattering amplitude that is obtained with the fitted potential listed in Table I.

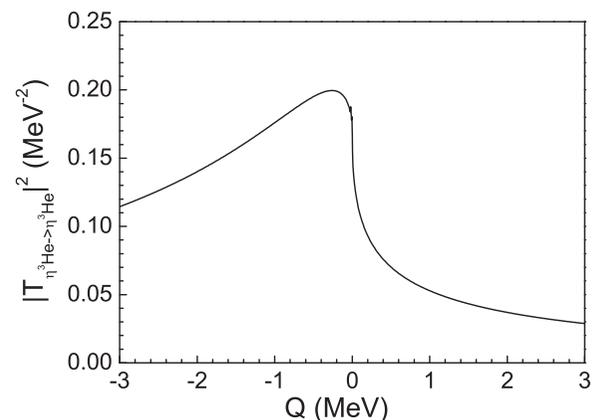


FIG. 7. Square of the absolute value of the $\eta^3\text{He} \rightarrow \eta^3\text{He}$ scattering amplitude.

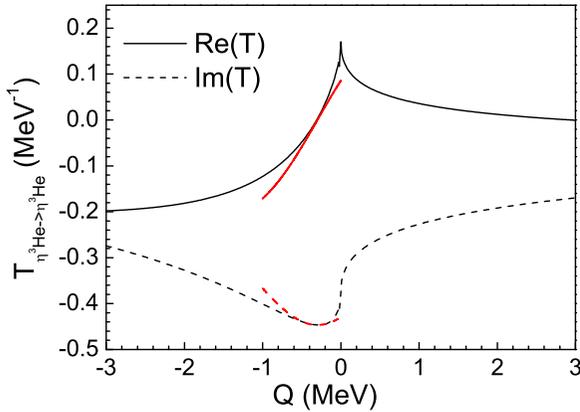


FIG. 8. Real and imaginary parts of the $\eta^3\text{He} \rightarrow \eta^3\text{He}$ amplitude T as a function of the excess energy Q . Red-solid and red-dashed curves are the real and imaginary parts of the Breit-Wigner structures of Eq. (15) fitted to the calculated amplitude.

window around -0.3 MeV this amplitude has the simple Breit-Wigner form with an energy M_R corresponding to a binding $B_E = 0.3 \pm 0.1$ MeV and a width $\Gamma = 3.0 \pm 0.5$ MeV. The errors quoted here are derived from those of the fitted potential listed in Table I.

The binding is determined from the zero of $\text{Re}(T)$ and the width from $\text{Im}(T)$. For this we write the ratio of the real and imaginary parts of T which, following Eq. (15), becomes

$$R = \frac{\text{Re}(T)}{\text{Im}(T)} = -\frac{2}{\Gamma}(Q + B_E), \quad (28)$$

where B_E is the binding energy (positive) of the $\eta^3\text{He}$ system. Our results for R shown in Fig. 9 allow us to estimate easily the values of B_E and Γ . The binding is very small and the width is much larger than the binding, in line with other theoretical studies of η bound in nuclei.

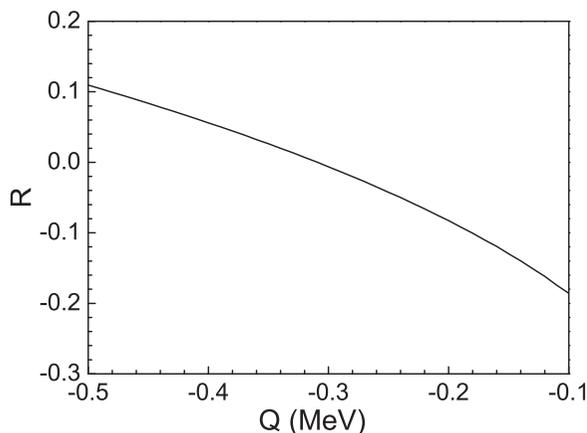


FIG. 9. Ratio of the real and imaginary parts of the $\eta^3\text{He} \rightarrow \eta^3\text{He}$ amplitude T as a function of the excess energy Q .

C. Systematic uncertainties

Two tests have been carried out to provide some indication of the systematic uncertainties and the stability of the results.

Although the nuclear density used to determine the G function of Eq. (10) is one of the important points of the analysis, we have tried a G function with a cut off of 300–400 MeV and conducted the fit again. It is well known that changes in G can be approximately absorbed in \tilde{V} to obtain the same T matrix. This is the case here, and we find that such large changes in G induce only changes of the order of 0.05 MeV in the binding energy and 0.4 MeV in the width.

Another test that we have undertaken is to assume an energy dependence of the parameters A and B that could arise from the basic in $pd \rightarrow \eta^3\text{He}$ production mechanism. For this we have taken a form $(1 + \gamma'Q)$ with γ' sufficiently small that for $Q < 10$ MeV the changes in $|A|^2$ and $|B|^2$ did not exceed 20%. The resulting changes in the binding energy and width were very small, being of the same order of magnitude as those induced by the changes in G . Compounding these systematic effects in quadrature, we conclude that

$$B_E = (0.30 \pm 0.10 \pm 0.08) \text{ MeV}, \quad (29)$$

$$\Gamma = (3.0 \pm 0.5 \pm 0.7) \text{ MeV}, \quad (30)$$

where the first errors are statistical and the second systematic.

VI. CONSIDERATIONS ON POLES OF THE AMPLITUDE

Let us suppose that a microscopic theory, where the decay channels are known, produces an amplitude like that in Eq. (15), which corresponds to a resonance. It is customary to assume that this amplitude has a pole at $\sqrt{s} = M_R - i\Gamma/2$, and this would be true if Γ were constant. In a dynamical theory this no longer follows. To see this, consider the case of an s -wave resonance where there is just one decay channel and Γ is proportional to the momentum p of the decay products in the resonance rest frame. If \sqrt{s} is made complex, then so is p and one does not find a solution to $\sqrt{s} - M_R + i\Gamma/2 = 0$. A solution may be obtained by changing $p \rightarrow -p$, which defines the second Riemann sheet.

However, when one has only an optical potential, as in the present case, one does not know the explicit channels contributing to the imaginary part and their strength and the different Riemann sheets are not defined. The only possibility to find poles is to change \sqrt{s} to a complex value, with the same optical potential, and look for a pole of the amplitude.

Imagine that we have an amplitude of the Breit-Wigner type of Eq. (15), valid in a range of values of \sqrt{s} , let us say $M_R \pm \Gamma$. In this case the amplitude has a pole at $\sqrt{s} = M_R - i\Gamma/2$. Since our amplitude behaves like a Breit-Wigner in a certain range of energies (see Fig. 8), we might think that it has a pole at $-B_E - i\Gamma/2$ [see Eqs. (29) and (30)]. However, for this to be true the range of the validity of the formula should stretch in an interval of about 6 MeV, when in reality it is barely valid in a range of 0.3 MeV. As a consequence, we do not find a pole at $-B_E - i\Gamma/2$ and, instead, we find it at $Q = (1.5 - i0.7)$ MeV, i.e., in the unbound region.

With the definition that a bound state corresponds to a pole of the amplitude below threshold, the potential that we obtain

would not produce a bound state. Solving the Schrödinger equation with bound-state boundary conditions would not have led to a solution since it would be equivalent to having a pole below threshold. This is indeed the situation of Ref. [23] where, for values of $\text{Re}(a'_{\eta N}) \simeq -0.5$ fm, as we found in Eq. (25), they would not obtain a bound state solution. We have checked that, by taking $\text{Im}(a'_{\eta N}) \simeq -0.25$ fm, we need $\text{Re}(a'_{\eta N}) \simeq -0.66$ fm in order to have a pole in the bound region at $Q = (-0.14 - i4.1)$ MeV. The model of Ref. [23] requires an even larger $\text{Re}(a'_{\eta N})$, of the order of 1 fm, to get bound states, but since they have a larger imaginary part of the potential, the resulting widths are of the order of 15 MeV. Given that the imaginary part of a potential acts rather like a repulsion, one can see consistency between their results and ours.

The issue of binding or not binding and the possibility to obtain this information from the $pd \rightarrow \eta^3\text{He}$ reaction has received much attention, and some theoretical works do predict a binding for $\eta^3\text{He}$ [48–51]. Some excitement was raised by a peak observed in the πp spectrum below the η threshold in the $\gamma^3\text{He} \rightarrow \pi^0 p X$ reaction [52], which was interpreted as evidence of a bound state. Even if the peak existed it would not be easy to distinguish between a bound state or a virtual state [53,54], but it was subsequently found that the structure observed was an artefact, related to the interplay of energy dependence and opening angle of the $\pi^0 p$ decay [34].

The analysis of the data in Ref. [53] was done on basis of the simplified amplitude of Eq. (27),² where the effective range and higher order terms were neglected, although their possible relevance in the problem was acknowledged in Ref. [55]. When using such a simplified amplitude, the question about bound versus unbound is easy to answer, and, as noted in Ref. [53], if $\text{Re}(a) > 0$ (in our notation) there is a bound state while if $\text{Re}(a) < 0$, there is a virtual state. We can see indeed that this approximation to f is insufficient in the present case, because we found $\text{Re}(a_{\eta^3\text{He}}) > 0$ [see Eq. (26)] but there is no pole below threshold, as Eq. (27) would suggest. This demonstrates the importance of going beyond the approximation of Eq. (27) for f , which is one of the accomplishments of the present work.

Regarding the sign of $a_{\eta^3\text{He}}$, most of the works done analyzing η production data close to threshold conclude that only $|a|$ can be obtained. Some attempts to obtain the sign were done in Ref. [55], where, again using the form of f in Eq. (27), early data on the $pd \rightarrow \eta^3\text{He}$ reaction were analyzed. With many assumptions and caveats, a scattering length of

$$a_{\eta^3\text{He}} = [(-4.3 \pm 0.3) - i(0.5 \pm 0.5)] \text{ fm} \quad (31)$$

(in our notation) was tentatively suggested. This encouraged the more precise $pd \rightarrow \eta^3\text{He}$ measurements that were used in the present study. The use of the new data [25,39], and the improved analysis presented here, determines the sign of the real part of $a_{\eta^3\text{He}}$ to be positive, as shown in Eq. (26).

The potential that we have found does not produce a pole below threshold and so, with the conventional definition, there

is no bound state. The search could stop at this point, but this would be premature since one then misses all the information that we have found about the amplitude below threshold, as shown in Figs. 7 and 8. If an experimentalist observed a structure of $|T|^2$ similar to that in Fig. 7 he or she would conclude that there was a bound state below threshold. From the practical point of view this may have more relevance than the existence of a pole in the unbound region and on a complex plane that is not experimentally accessible. The fact that, with the potential that we have derived, there is a pole in the unbound region might lead one to think that there is a resonance around that pole position. This would be the wrong conclusion. We consider that it is more appropriate to study the peak structure of the amplitude below threshold, and this is the attitude we have taken here.

VII. SUMMARY AND CONCLUSIONS

We have performed an analysis of data on the $pd(dp) \rightarrow \eta^3\text{He}$ reaction close to threshold. These consist of total cross sections and angular asymmetries up to an excess energy of 10 MeV. Unlike former approaches that make a parametrization of the amplitude, we express the observables in terms of an optical potential from which the $\eta^3\text{He}$ scattering amplitude is evaluated. The T matrix is evaluated from the potential using the Bethe-Salpeter equation and the loop function G of the intermediate $\eta^3\text{He}$ state. This reflects the range of the $\eta^3\text{He}$ interaction, as given by the empirical density of the ^3He nucleus. The results lead to a structure of the T matrix that is quite different from the usual parametrizations of the data.

The potential and other parameters related to the production vertices are fitted to the data and in this way we deduce that there is a weakly bound $\eta^3\text{He}$ state with binding energy of the order of 0.3 MeV and a width of the order of 3 MeV. We also obtain an $\eta^3\text{He}$ scattering length of the order of $(2.2 - i4.9)$ fm. It is important to note that the fit in terms of the potential resolves an ambiguity in the sign of the real part of $a_{\eta^3\text{He}}$ that was present in previous analyses.

In summary, the new approach to the analysis of the $pd \rightarrow \eta^3\text{He}$ data close to threshold has proved quite useful and has been able to provide information on a bound state of the $\eta^3\text{He}$ system. This agrees with previous theoretical work that the width of the bound state should be significantly larger than the binding. A peak in $|T|^2$ is predicted below threshold and this might in principle be detected experimentally in some breakup channel of the pd system. However, one must anticipate that the background from other sources, where the bound state is not produced, could be very large and obscure any signal.

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²There are two different sign conventions for the scattering length and we are using the one where $k \cot(\delta) \simeq -\frac{1}{a} + \frac{1}{2}r_0k^2$. In this notation, unitarity requires that $\text{Im}(a) \leq 0$.

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