Introduction of Total Variation regularization into Filtered Back-projection algorithm

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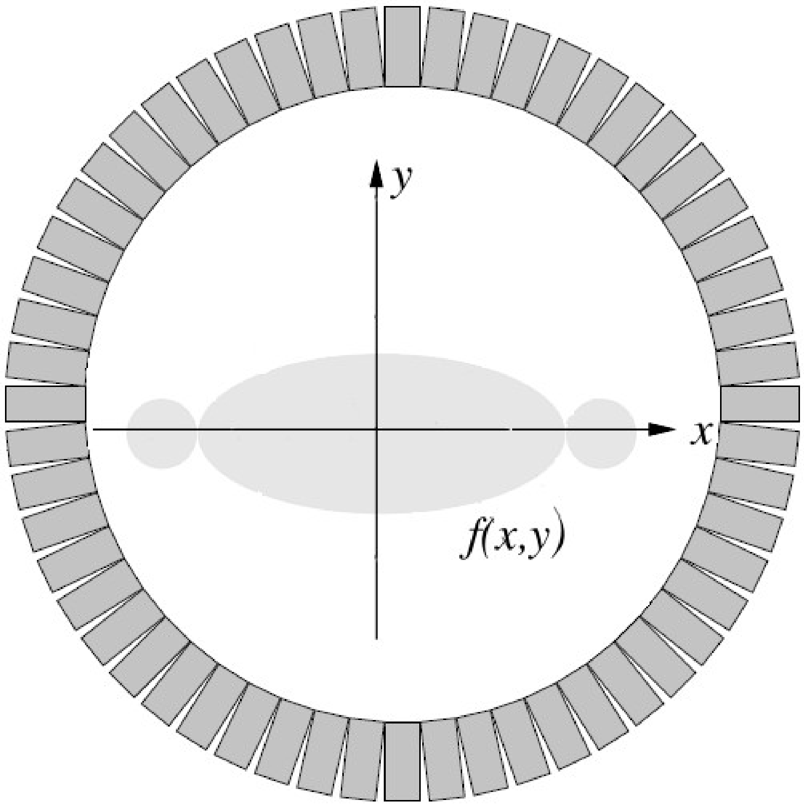
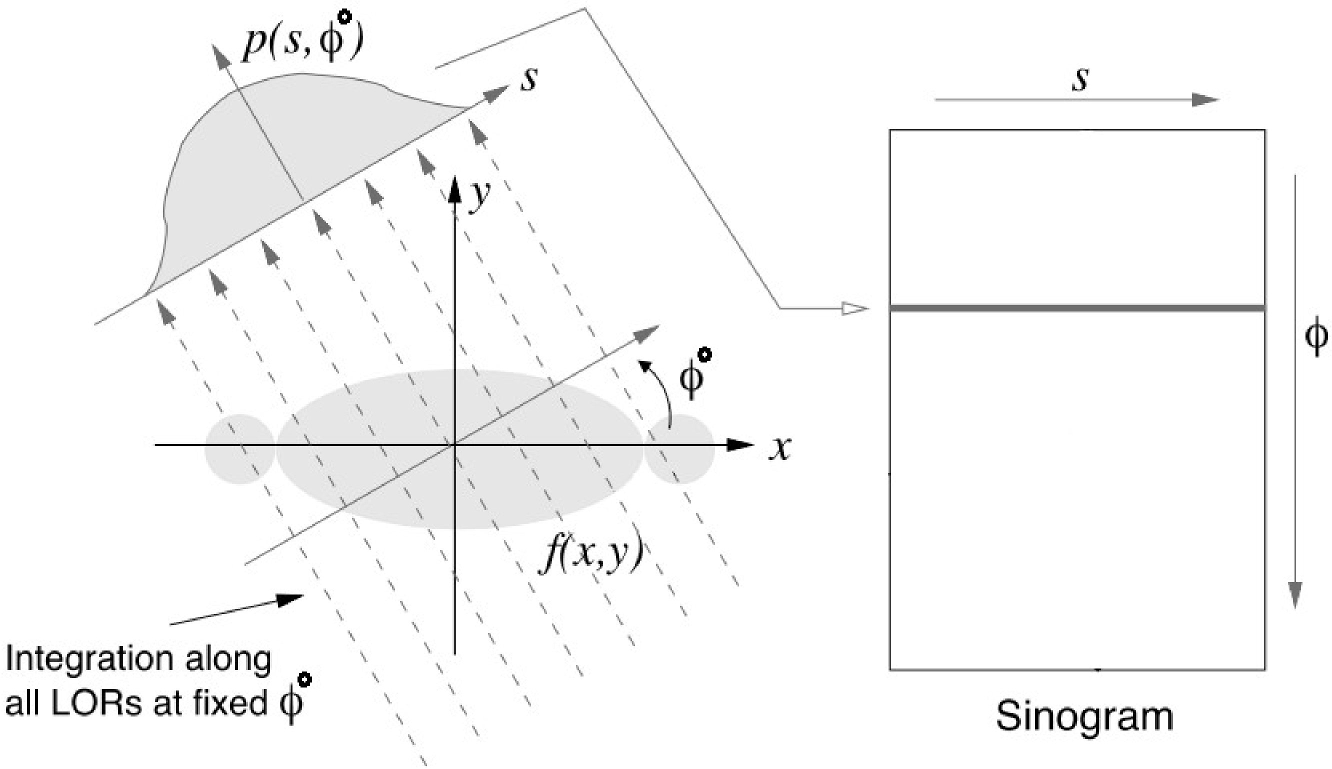
Kraków, 07.06.2017

1. Introduction



- PET scanner forms a ring in 2-dimensional plane (x,y)
- Filtered back-projection (FBP) was the first PET reconstruction method

1. Introduction



- A single projection p is formed from integration along all LORs at fixed angle
- The collection of projections for all angles forms a 2-dimensional sinogram

2. Filtered Back-projection (FBP)

Filtered Back-projection

- Registration of sinogram
- Filtration (in projection domain)

$$p^F(s,\phi) = \mathcal{F}_1^{-1}\left\{W(\upsilon_s)\left|\upsilon_s\right|\mathcal{F}_1\left\{p(s,\phi)\right\}\right\}$$

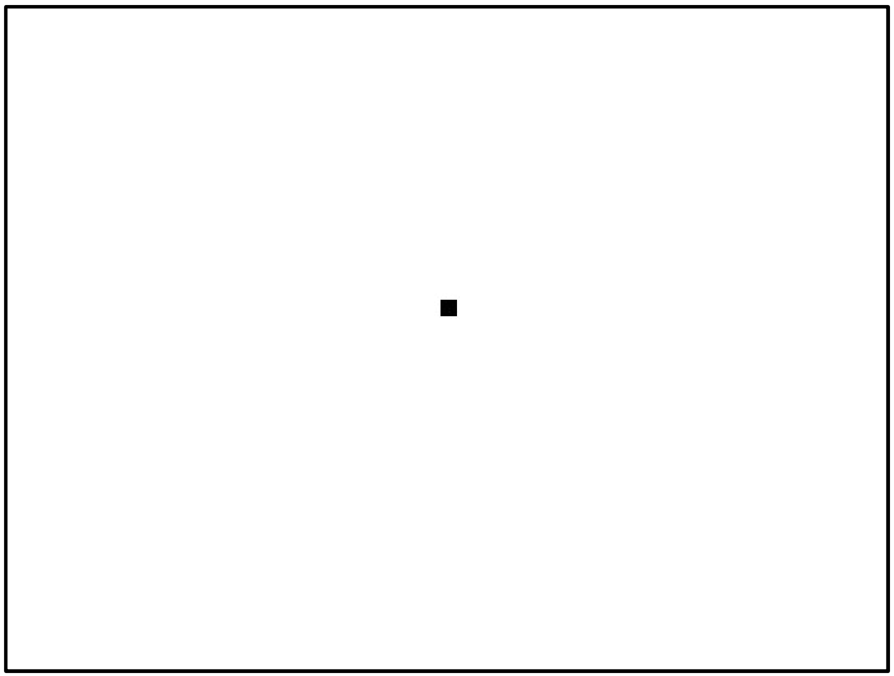
- Back-projection to image domain

$$f(x,y) = \int_0^{\pi} p^F(s,\phi)\,d\phi$$

2. Filtered Back-projection (FBP)

Filtered Back-projection	**Back-projection Filtration**
• Registration of sinogram	Registration of sinogram
↓	↓
• Filtration (in projection domain)	Back-projection to image domain
$p^F(s,\phi) = \mathcal{F}_1^{-1}\{W(\upsilon_s)\lvert\upsilon_s\rvert\mathcal{F}_1\{p(s,\phi)\}\}$	$b(x,y) = \int_0^\pi p(s,\phi)\,d\phi$
↓	↓
• Back-projection to image domain	Filtration (in image domain)
$f(x,y) = \int_0^\pi p^F(s,\phi)\,d\phi$	

3. Back-projection filtration (BPF)



$f(x,y)$

Back-projection Filtration

Registration of sinogram

↓

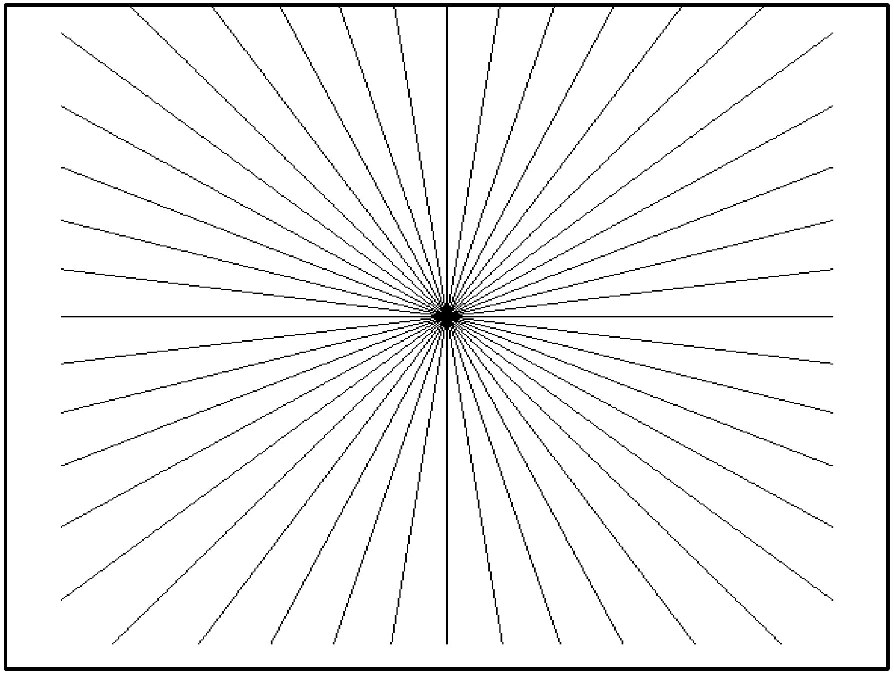
Back-projection to image domain

$$b(x,y) = \int_0^{\pi} p(s,\phi)\,d\phi$$

↓

Filtration (in image domain)

3. Back-projection filtration (BPF)



Back-projection Filtration

Registration of sinogram

↓

Back-projection to image domain

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↓

Filtration (in image domain)

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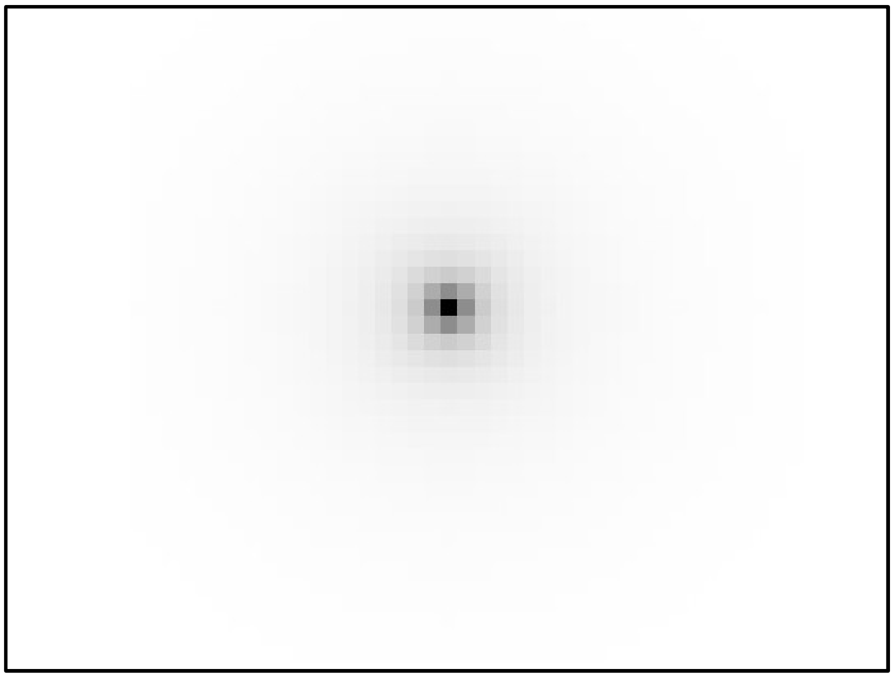


$$b(x,y) = (h * f)(x,y) = h(x,y)$$

Back-projection Filtration

Registration of sinogram

↓

Back-projection to image domain

$$b(x,y) = \int_0^{\pi} p(s,\phi)\, d\phi$$

↓

Filtration (in image domain)

3. Back-projection filtration (BPF)

The impulse response h is equal to:

$$h(x,y) = \left(x^2 + y^2\right)^{-1/2}$$

Therefore, we search for the solution $\hat{f}$ given:

$$b(x,y) = (h * f)\,(x,y)$$

of equivalently in matrix notation:

$$b = Af.$$

Back-projection Filtration

Registration of sinogram

↓

Back-projection to image domain

$$b(x,y) = \int_0^{\pi} p\,(s,\phi)\,d\phi$$

↓

Filtration (in image domain)

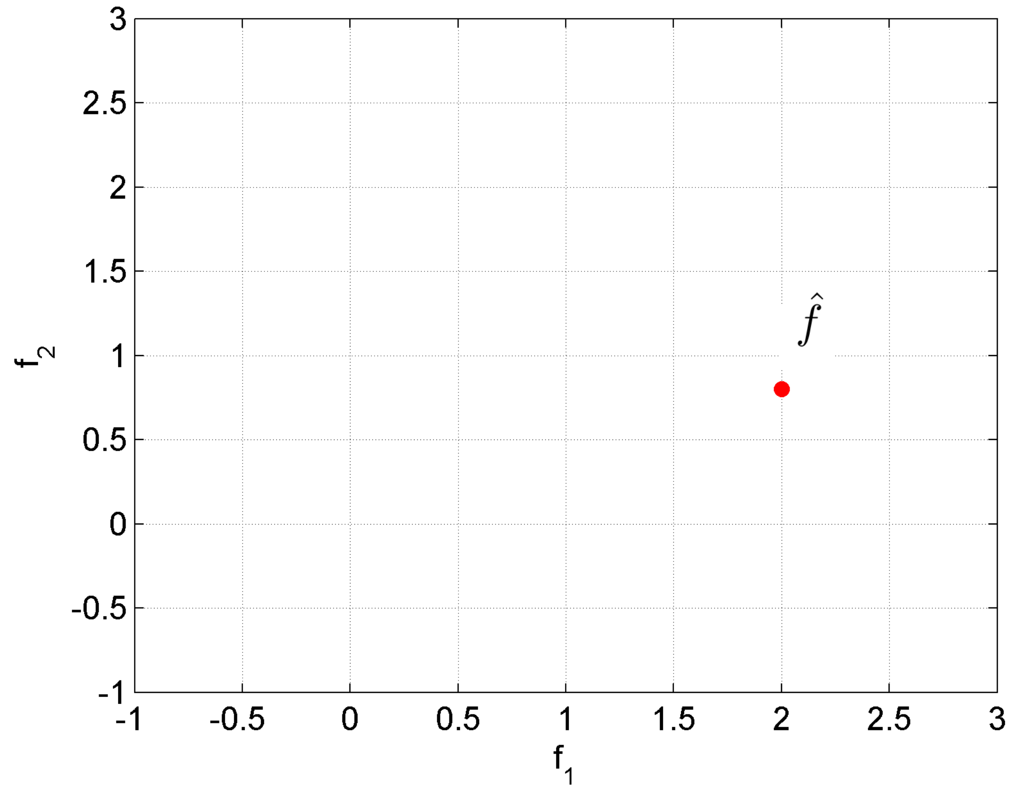
4. Total Variation regularization

The direct solution of the linear problem

$$b = Af$$

may be considered as the least squares minimization:

$$\hat{f} = \arg\min ||b - Af||_2^2.$$



4. Total Variation regularization

The direct solution of the linear problem

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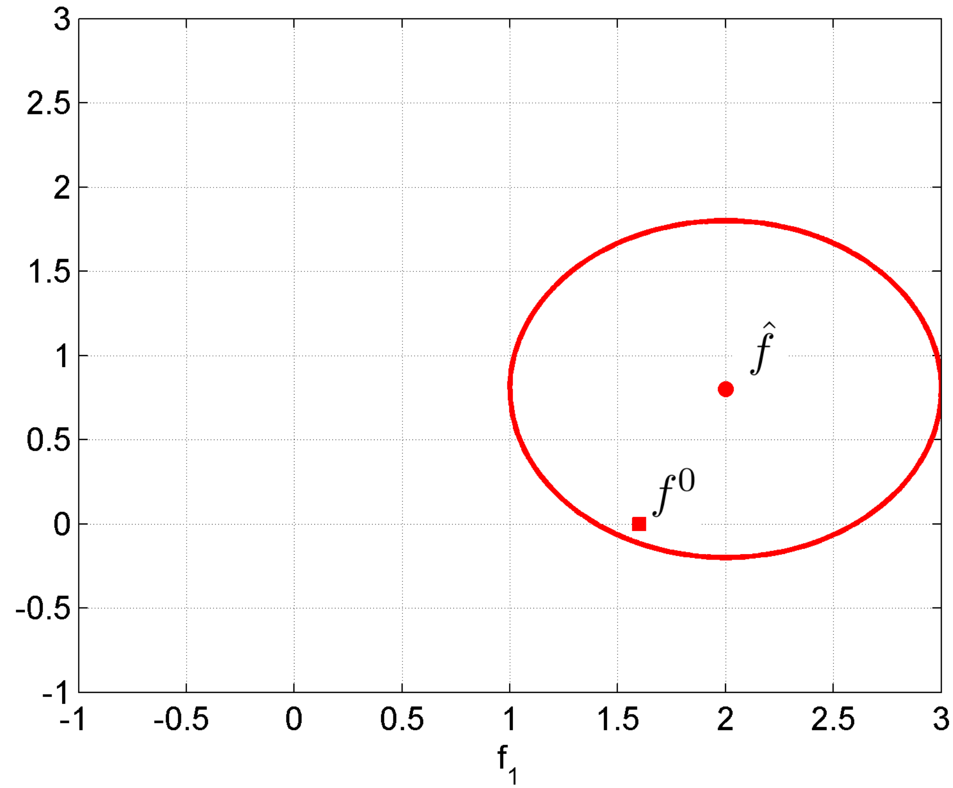
may be considered as the least squares minimization:

$$\hat{f} = \arg\min ||b - Af||_2^2.$$

In fact the true solution f^0 lies in the area

$$||b - Af||_2^2 \leqslant \epsilon,$$

where ϵ is the size of the error term.



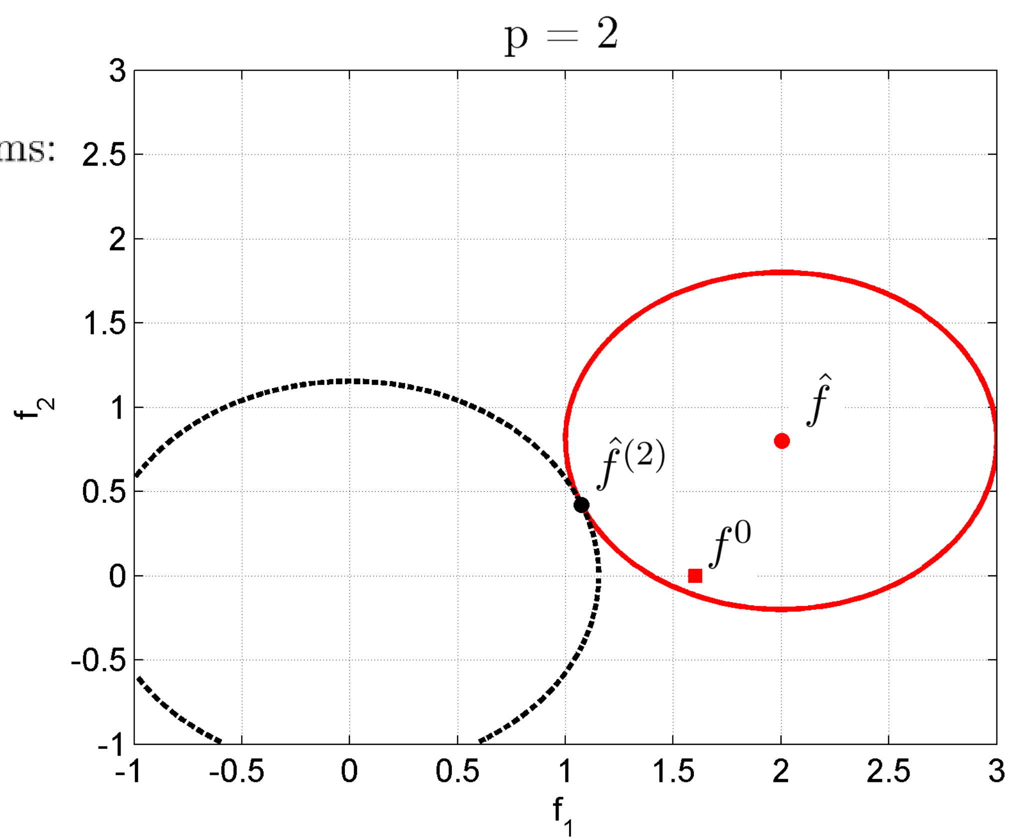
4. Total Variation regularization

Searching for the sparse solution $\hat{f}^{(p)}$ of linear system $b = Af$ with different p norms:

$$||f||_p = \left(\sum_i^N |f_i|^p \right)^{1/p}$$

The optimization task is:

$$\min \quad ||f||_p \quad \text{subject to} \quad ||b - Af||_2^2 \leqslant \epsilon.$$



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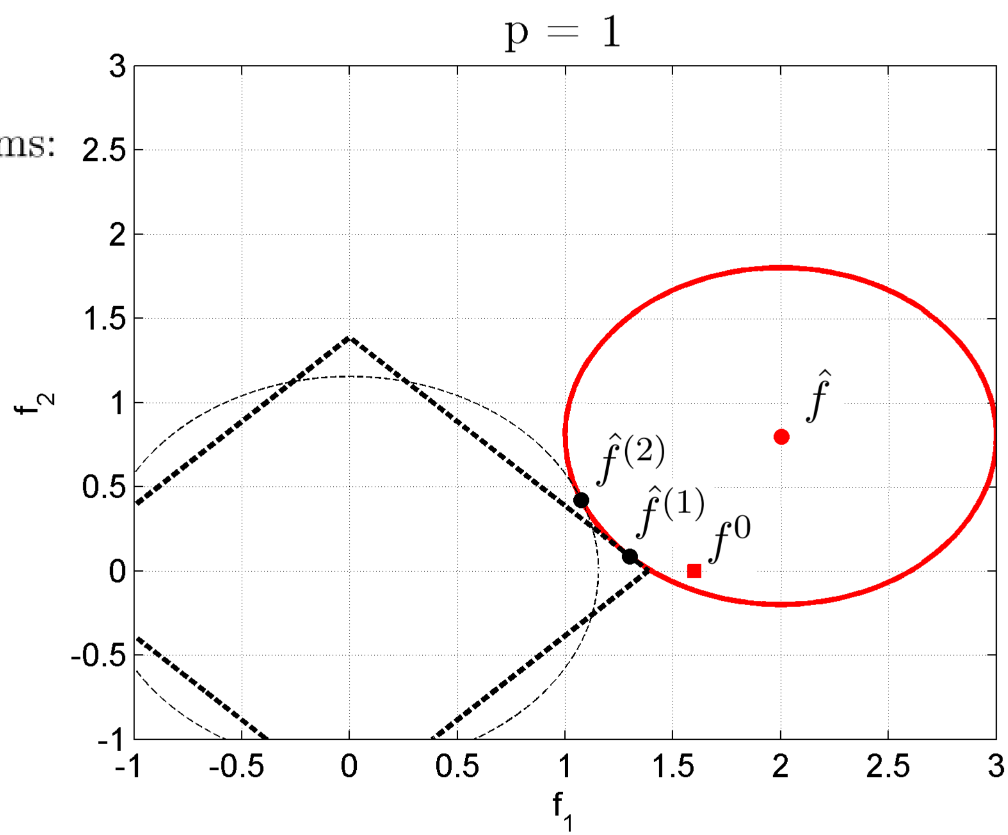
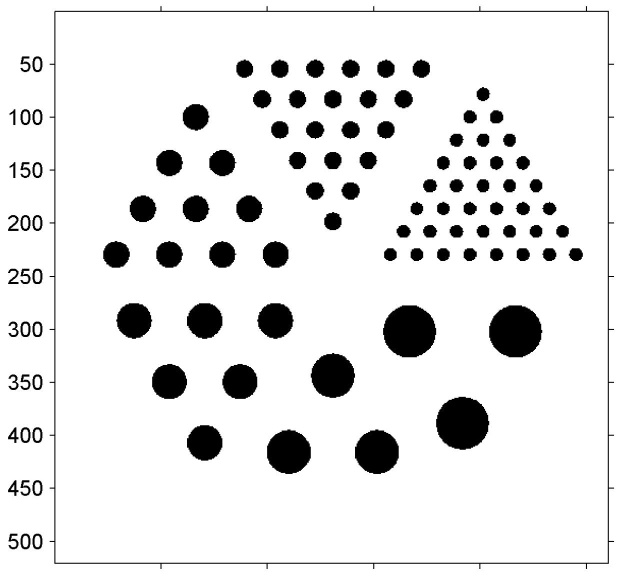


It can be shown that for $p \leqslant 1$ sparse solution might be found.

4. Total Variation regularization

- Typical PET image *f* cannot be regarder as a sparse image
- The transformation that enhance the sparsity has to applied



f

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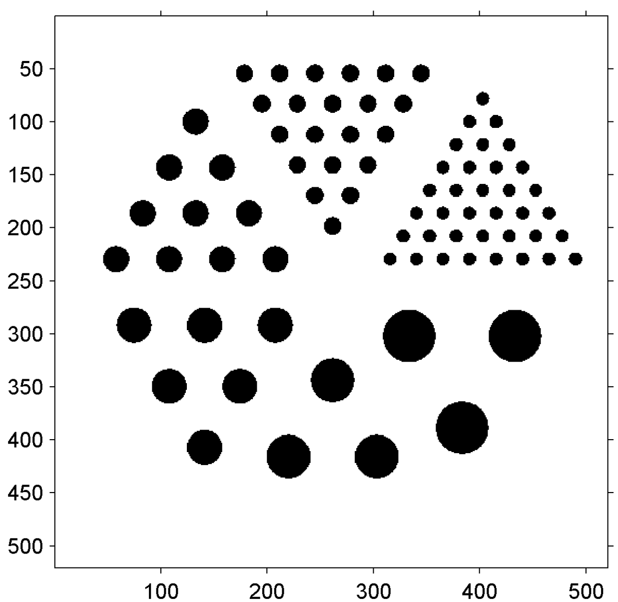
4. Total Variation regularization

- Typical PET image *f* cannot be regarder as a sparse image
- The transformation that enhance the sparsity has to applied
- An alternate model is that the gradient is sparse (*D* is gradient operator)



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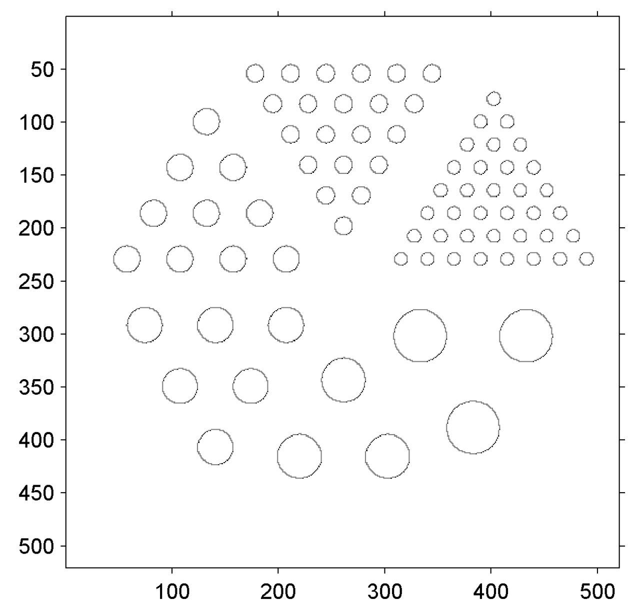

Df

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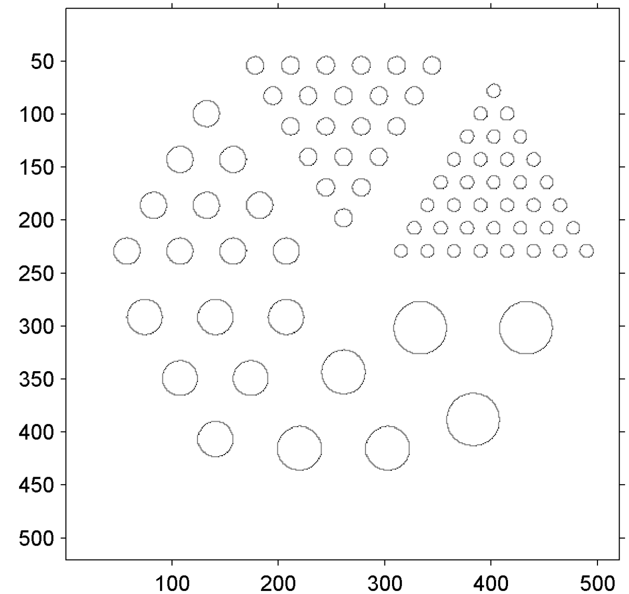
4. Total Variation regularization

The Total Variation (TV) of f is the sum of the magnitudes of discrete gradient at every point:

$$\mathrm{TV}(f) = \sum_i ||D_i f||_1$$
$$= ||Df||_1$$

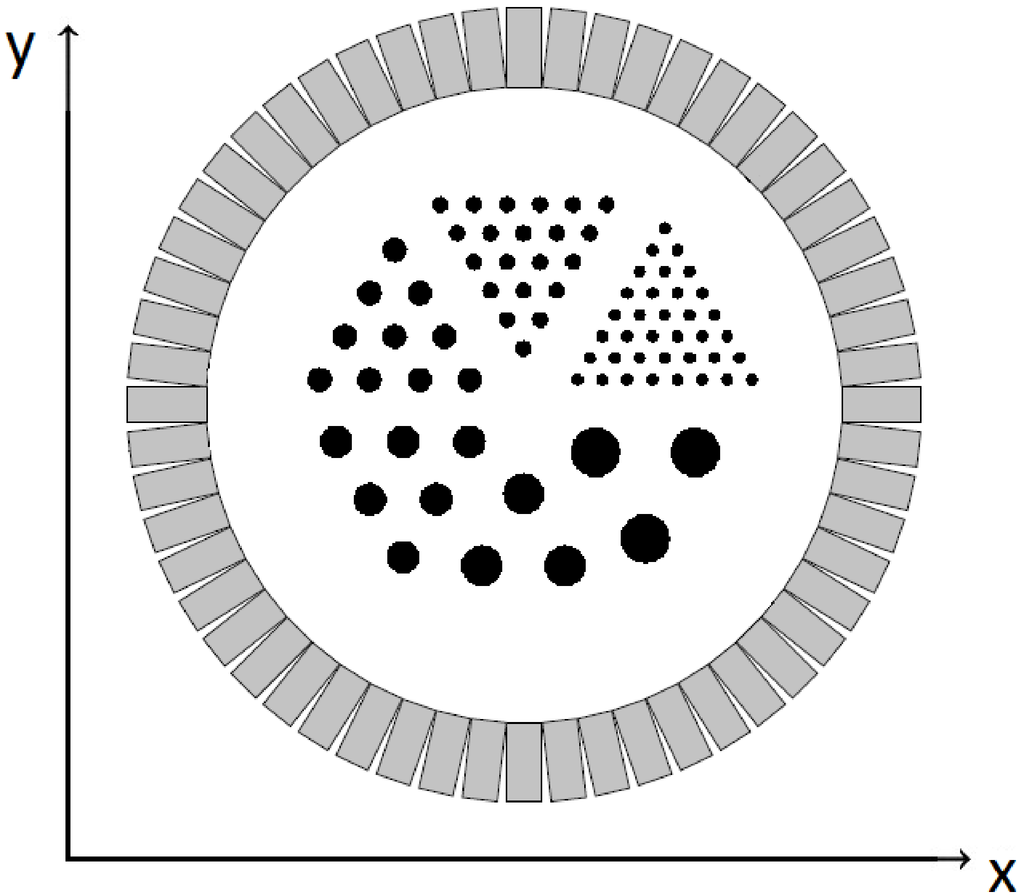
Searching of the solution $\hat{f}$ with TV norm is an optimization task:

$$\min \quad \mathrm{TV}(f) \quad \text{subject to} \quad ||b - Af||_2^2 \leqslant \epsilon.$$



Df

5. Results

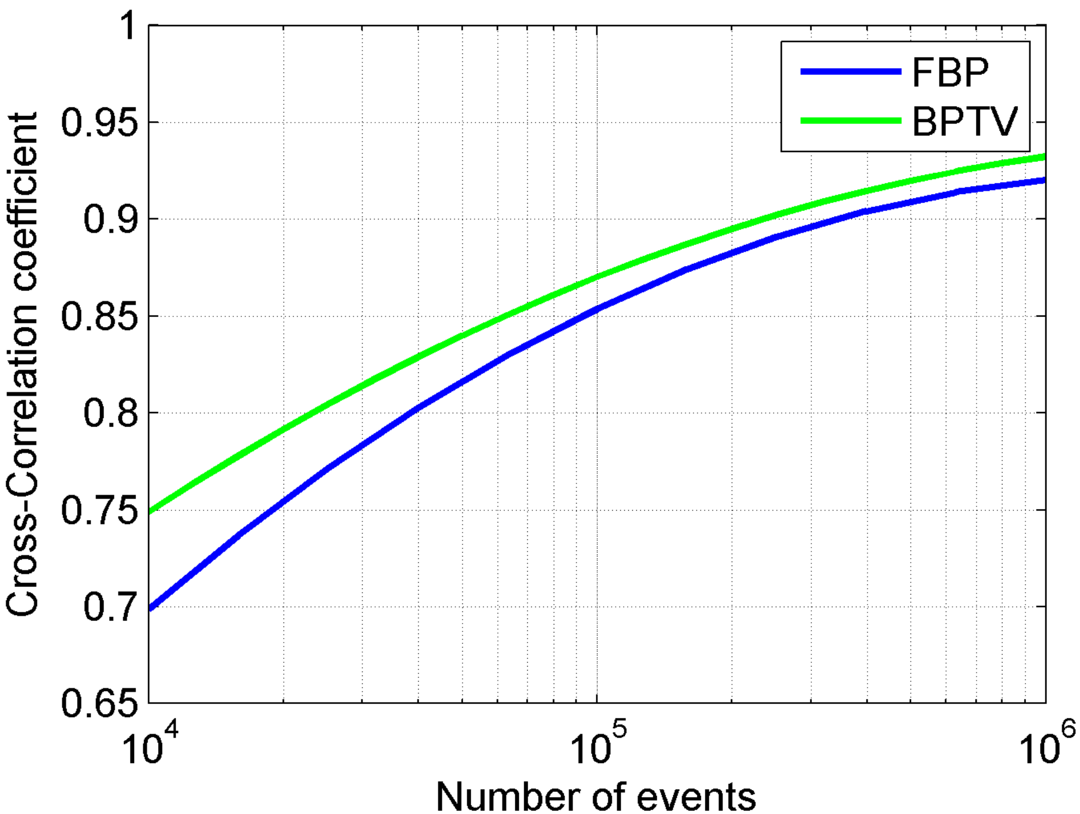


Geometry:
- 1 layer with **384** strips,
- 84 cm diameter,

Derenzo phantom with
1 x 1 mm2 pixel size,

The rods are grouped into six sets with diameters of:
10, 15, 23, 32, 40, 48 mm.

5. Results

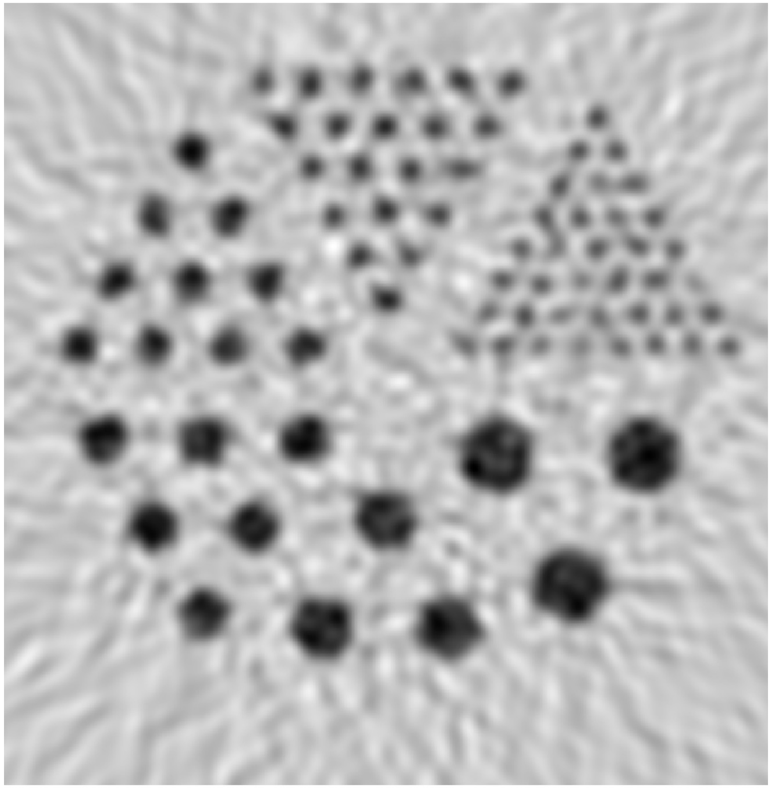


- FBP and BPTV were optimized in the sense of choosing the best regularization parameters that maximize the correlation coefficient
- Experiments have been performed for different number of registered events, equivalently LORs
- The only difference between FBP and BPTV concerns the regularization approach

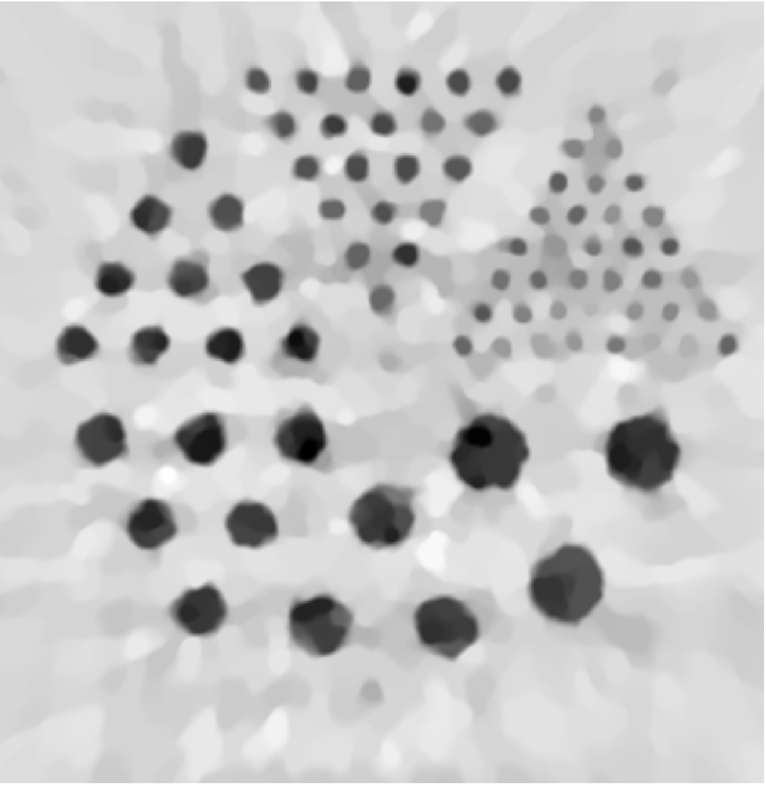
*BPTV (Back-projection Total Variation regularization)

5. Results

FBP (BPF)



BPTV



- Comparison of the performance of the FBP and BPTV algorithms
- The number of registered events (LORs) equal 100 000

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6. Summary

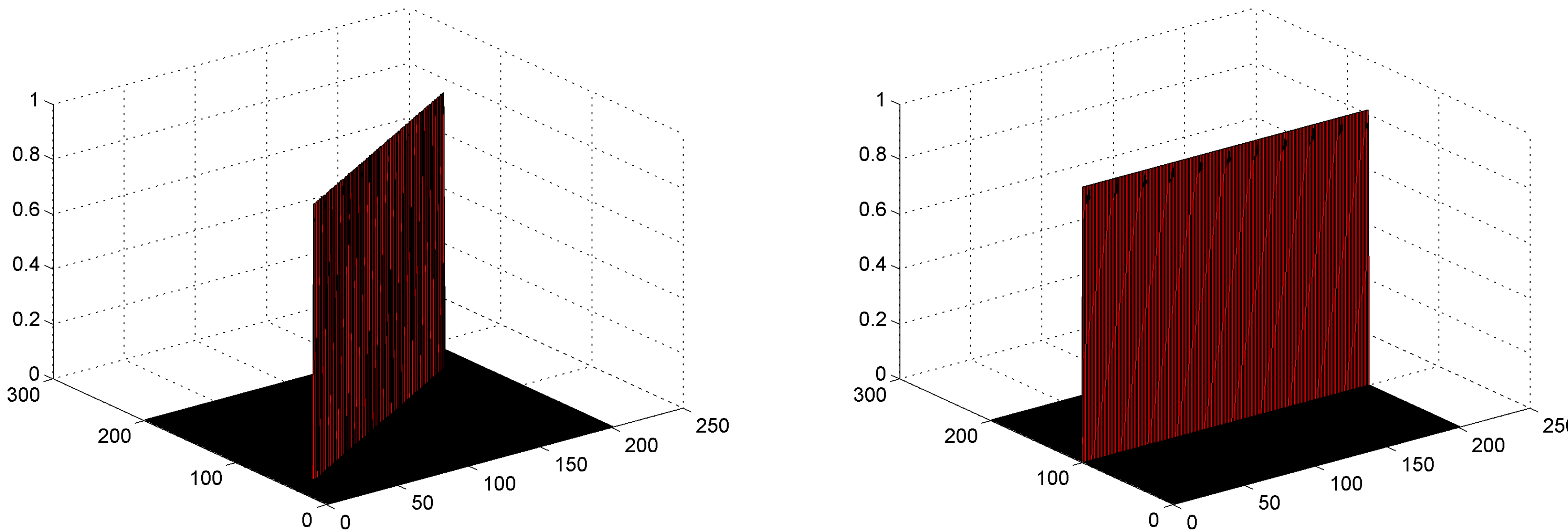
Total Variation regularization is a proper approach if the gradient of the underlying image is sparse

It was demonstrated that using the Total Variation regularization improves the quality of reconstructed image

The processing in image space has more benefits; there is an easy way to include the information from Time-of-Flight (TOF) measurement

$$\min \quad \mathrm{TV}(f) \quad \text{subject to} \quad ||b - Af||_2^2 \leqslant \epsilon.$$

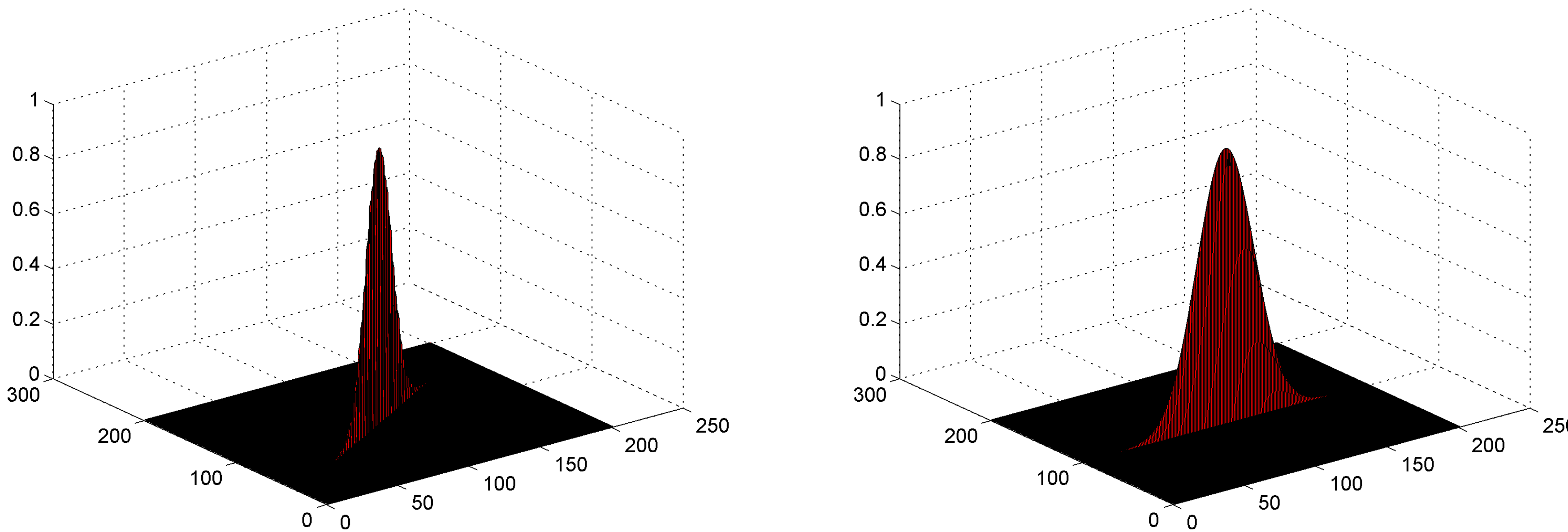
Introduction of Time of Flight (TOF)



$$h(x,y) = \left(x^2 + y^2\right)^{-1/2}$$

- Consider a function f(x,y) describing point source activity
- Calculation of the impulse response (*h*) for non-TOF measurement

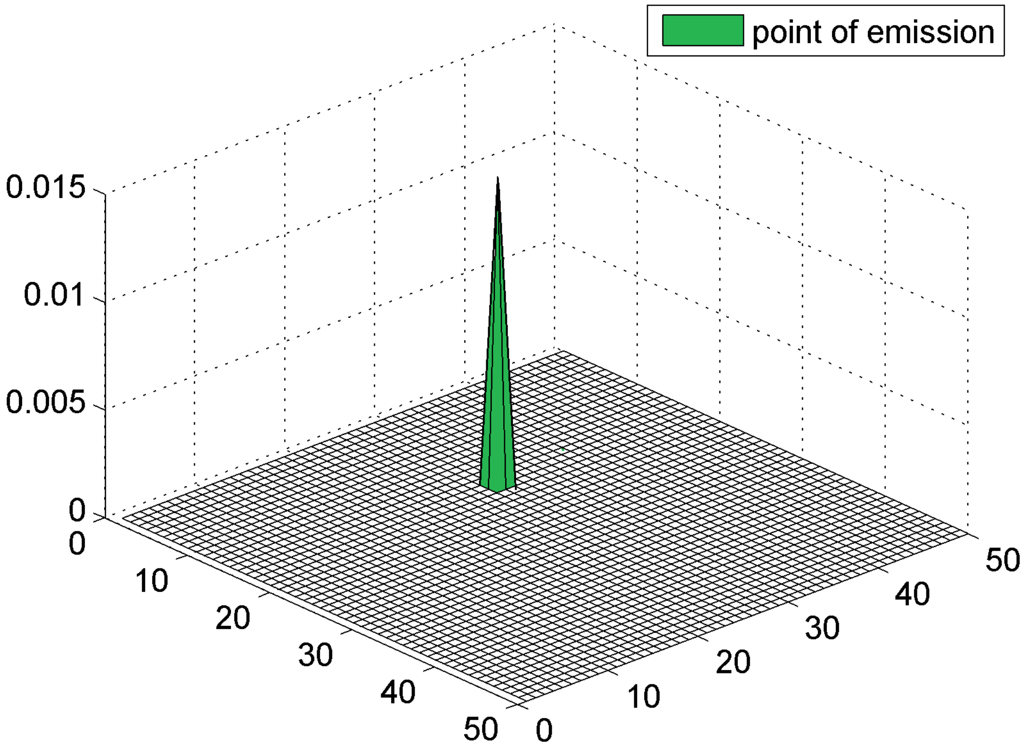
Introduction of Time of Flight (TOF)



$$h_{\mathrm{TOF}}(x,y) = h(x,y) \cdot e^{-\frac{x^2+y^2}{2\sigma(r)}}$$

- Time information from both detectors leads to better localisation on LOR
- The impulse response *h* and therefore matrix *A* need to be recalculated

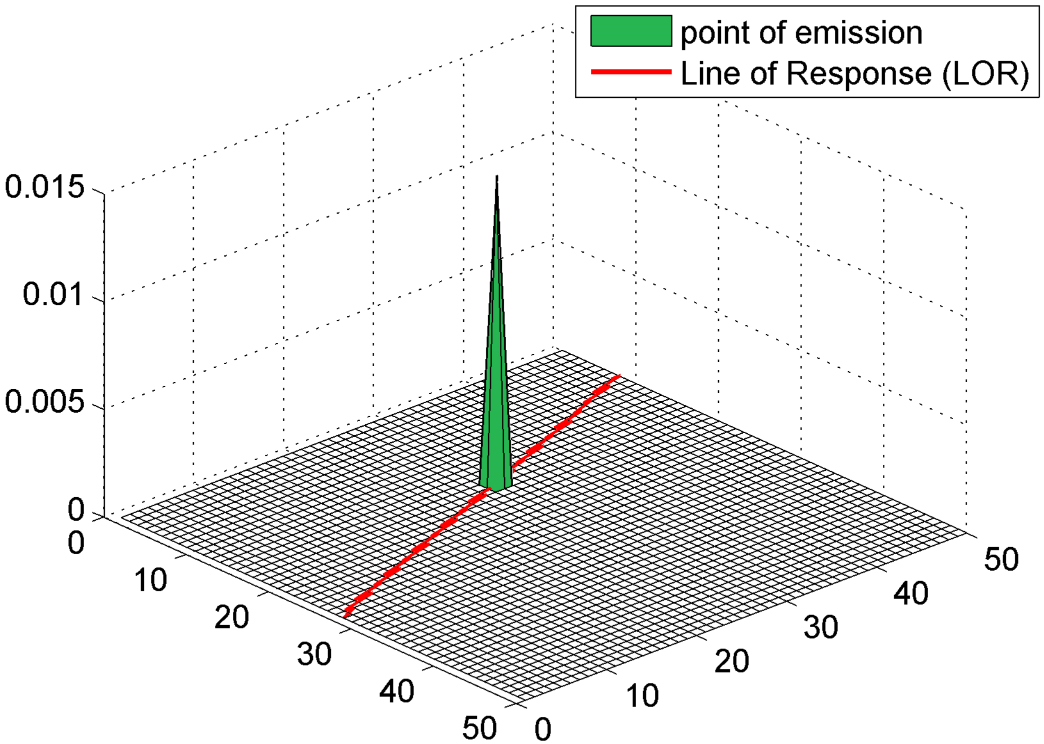
Introduction of Time of Flight (TOF)



- We need to calculate image b in other way:

$$\min \quad \mathrm{TV}(f) \quad \text{subject to} \quad ||b - Af||_2^2 \leqslant \epsilon.$$

Introduction of Time of Flight (TOF)



- Consider a point of emission
- The measurement was provided by a pair of detectors at the ends of red line

$$\min \quad \mathrm{TV}(f) \quad \text{subject to} \quad ||b - Af||_2^2 \leqslant \epsilon.$$

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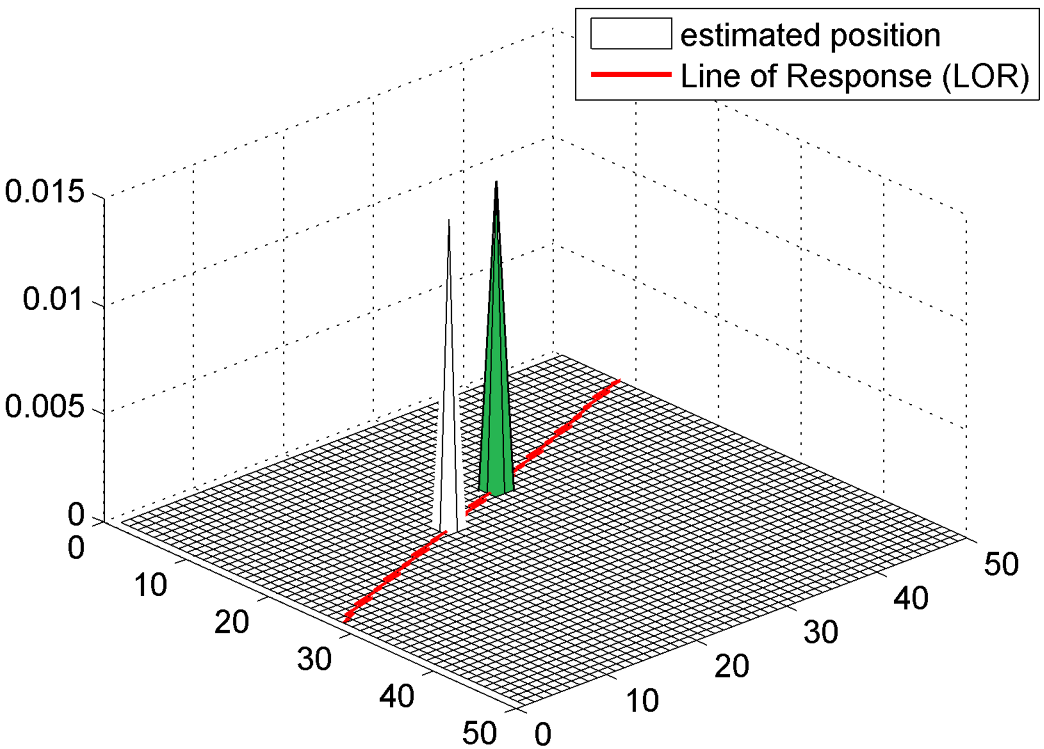
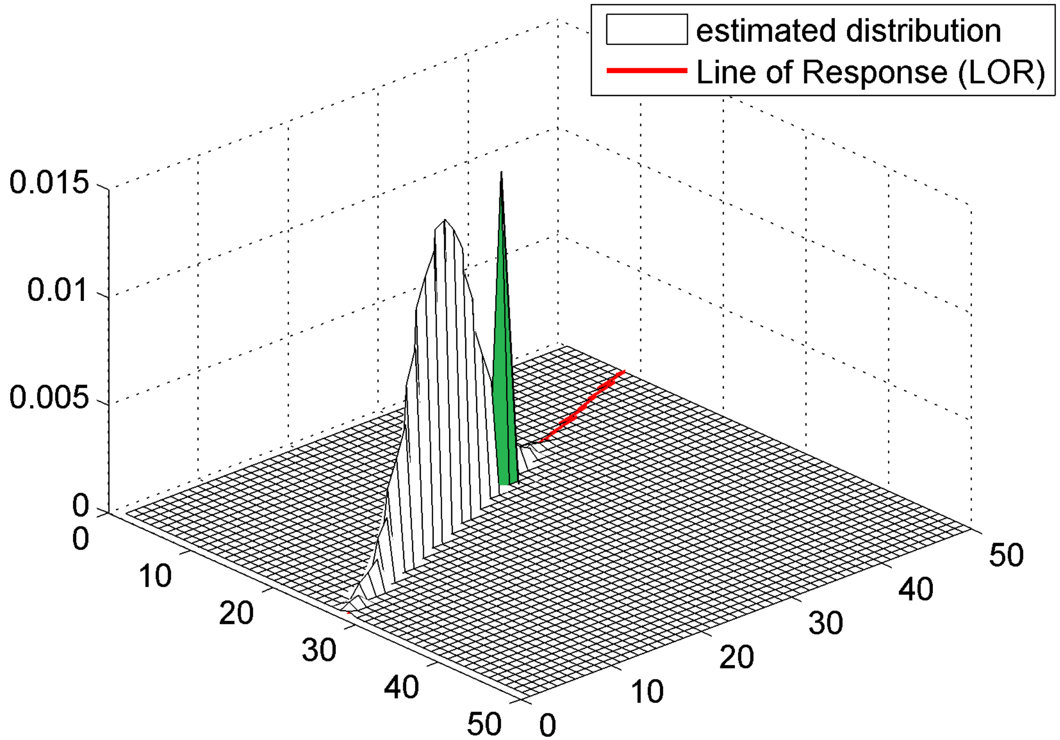
Introduction of Time of Flight (TOF)



- The information about time allows us to estimate the position along LOR

$$\min \quad \mathrm{TV}(f) \quad \text{subject to} \quad ||b - Af||_2^2 \leqslant \epsilon.$$

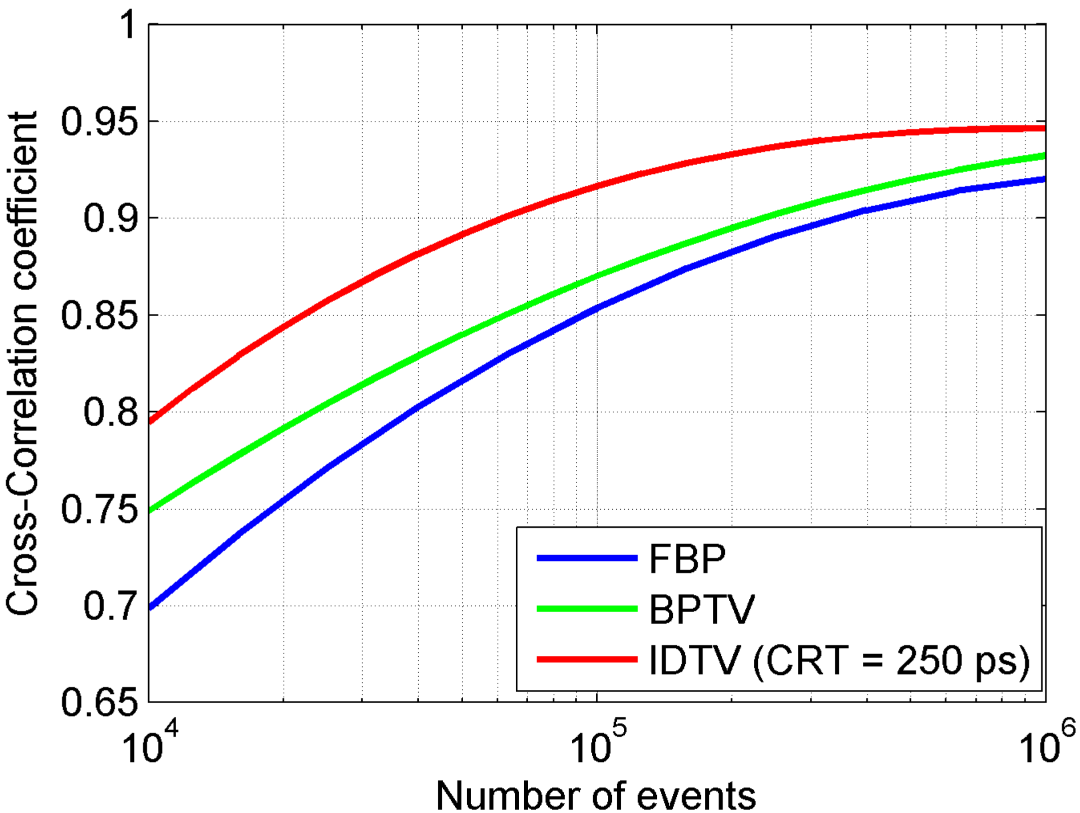
Introduction of Time of Flight (TOF)



- The information about time allows us to estimate the position along LOR
- We may estimate the distribution along LOR and include all points around the estimated position
- The repetition of this procedure for all registered events generates the blurred image *b*

$$\min \quad \mathrm{TV}(f) \quad \text{subject to} \quad \|b - Af\|_2^2 \leqslant \epsilon.$$

Introduction of Time of Flight (TOF)



- All methods were optimized in the sense of choosing the best regularization parameters that maximize the correlation coefficient
- IDTV method takes into account TOF information

*IDTV (Image-domain Total Variation regularization)