THREE-PHOTON ENTANGLEMENT FROM POSITRONIUM REVISITED

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A layman's view

1 A quick view at orthopositronium: matrix element, kinematics, distributions etc.

2 Three photon entanglement from orthopositronium
GHZ versus the W-class
classification according to spin projection
use of the full phase space circular, linear and linear transverse/parallel polarizations

3 Correlations (iconoclastic)
The two level formalism in two dimensions versus two level formalism in three dimensions (photon)

4 Clauser-Horne-Shimony inequalities for three photons

Motivation: An interplay between relativistic dynamics and kinematics and quantum entanglement

Not to forget: The CRACOW J-PET where apart from medical applications a study of the three photon entanglement from positronium could be done.

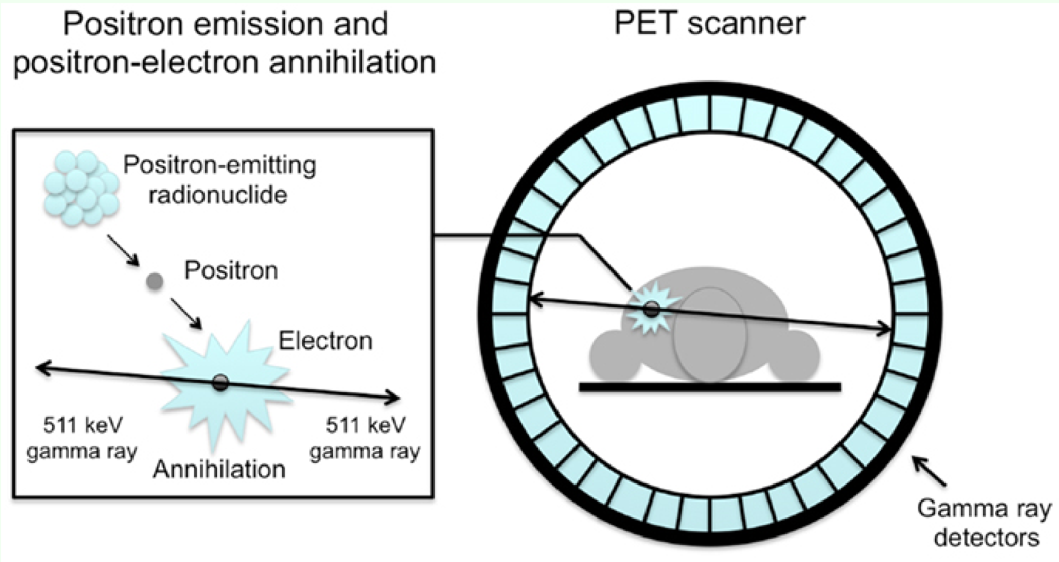


Figure 1: Positron electron tomography uses also photons from positronium (Cracow J-PET)

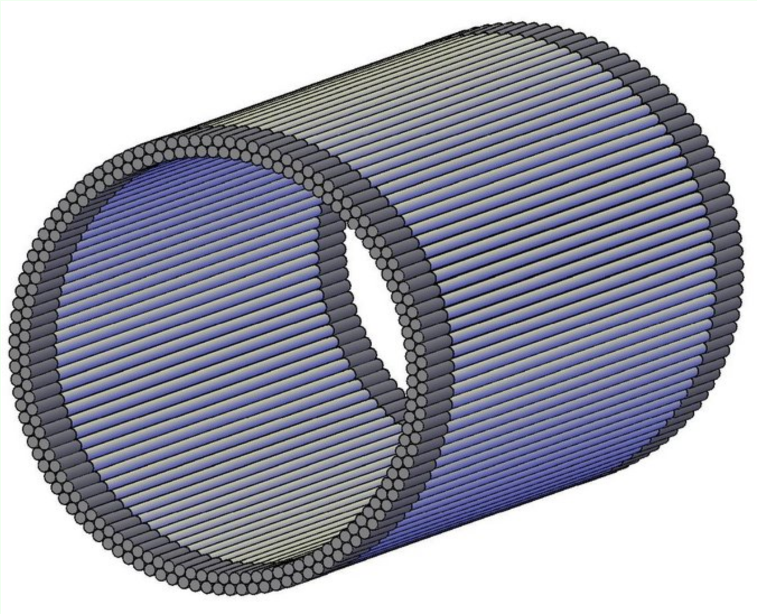


Figure 2: “A 4π coverage” of the photons (Cracow J-PET)

Figure 3: A purely leptonic hydrogen-like atom

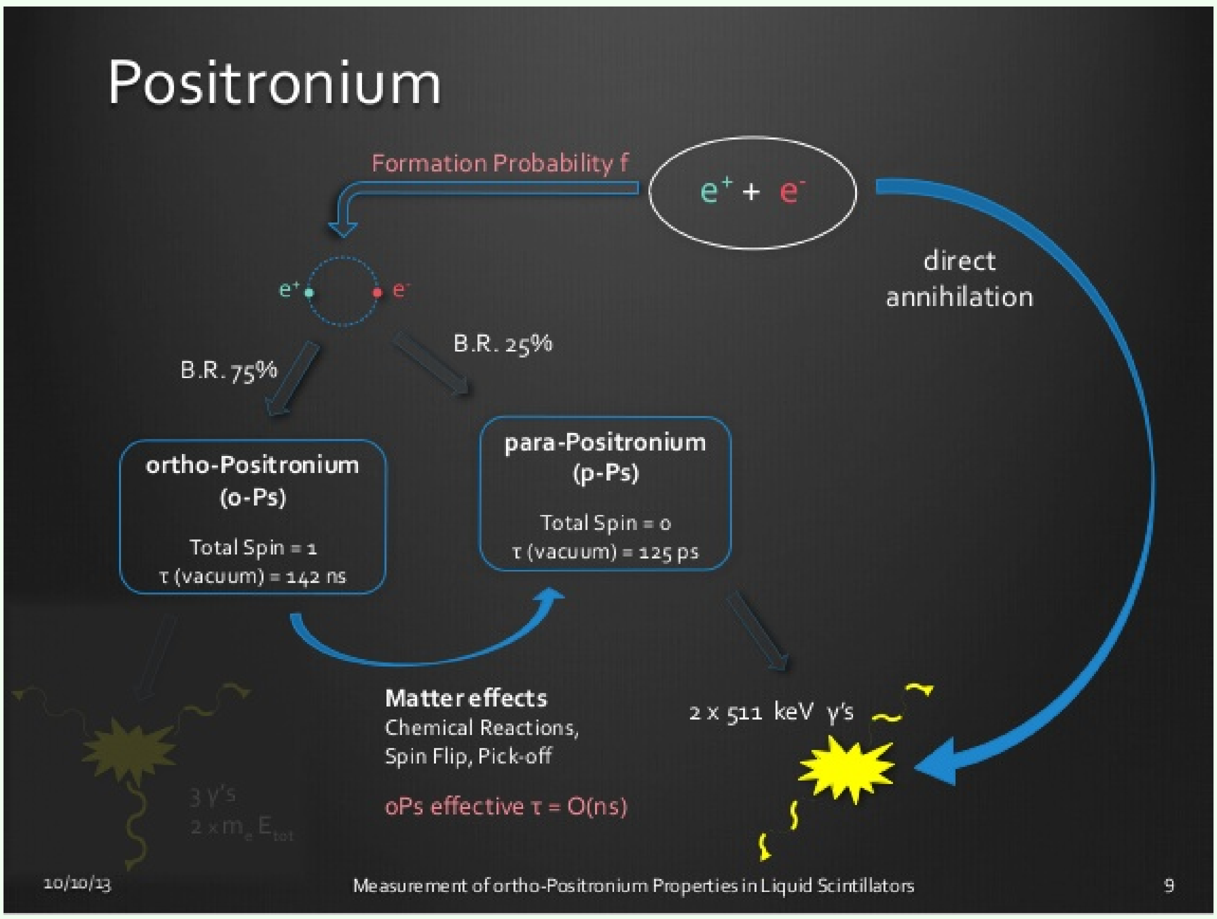


Figure 4: Decays into two or three photons (by annihilation) depending on the spin of the positronium

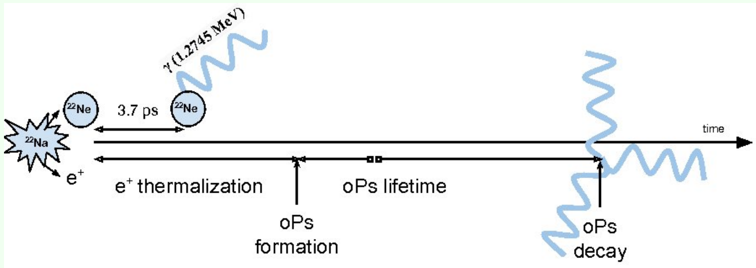


Figure 5: Its production uses the positron from nuclear proton decay.

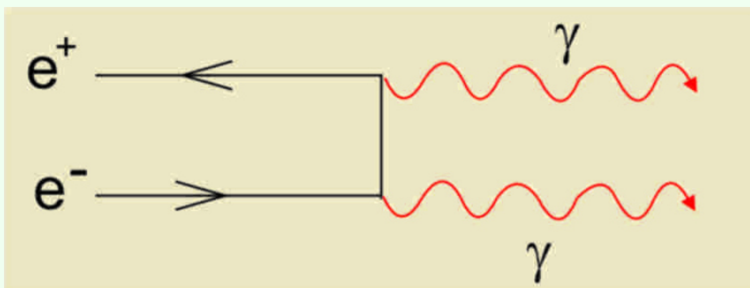


Figure 6: The parapositronium $S = 0$ decays into two photons (entangled)

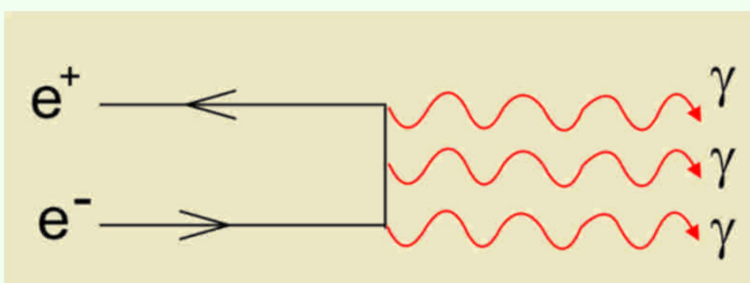


Figure 7: The orthopositronium $S = 1$ decays into three photons (entangled)

$$(e^+e^-)_{J^{PC}=1^{--}} \to \gamma_1 + \gamma_2 + \gamma_3$$
$$\gamma_i = \gamma[\boldsymbol{\epsilon}(\hat{\mathbf{k}}_i, \lambda), \mathbf{k}_i]$$

For circular polarization we have the polarization vectors

$$\begin{aligned} \boldsymbol{\epsilon}(\hat{\mathbf{k}}_i, \lambda) &= -\frac{\lambda}{\sqrt{2}}\Big(\cos\theta_i \cos\Phi_i - i\lambda \sin\Phi_i, \cos\theta_i \sin\Phi_i \\ &+ i\lambda\cos\Phi_i, -\sin\theta_i\Big) \end{aligned} \quad (1)$$

with $\lambda = \pm$ and the angles are of

$$\hat{\mathbf{k}}_i = (\cos\Phi_i \sin\theta_i, \sin\Phi_i \sin\theta_i, \cos\theta_i)$$

Kinematics: energy-momentum conservation

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$$

$$k_1 + k_2 + k_3 = m \simeq 2m_e$$

The momentum conservation defines a plane in which the photons move. The plane changes its orientation from event to event.

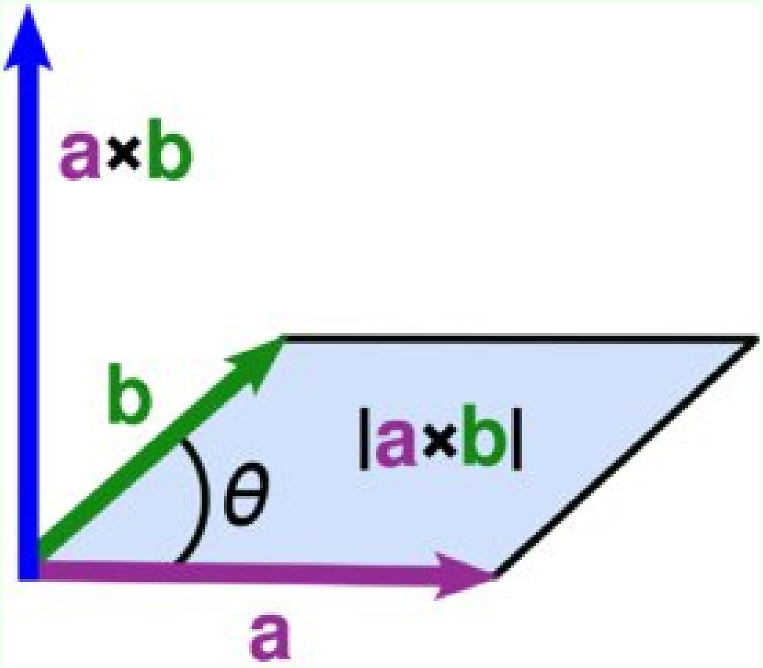


Figure 8: We can choose $\mathbf{a} = \mathbf{k}_1$ and $\mathbf{b} = \mathbf{k}_2$

The unit vector

$$\hat{\mathbf{n}} = \frac{\mathbf{k}_1 \times \mathbf{k}_2}{|\mathbf{k}_1 \times \mathbf{k}_2|}$$

is perpendicular to the plane. Often one chooses

$$\hat{\mathbf{n}} = \hat{\mathbf{z}}$$

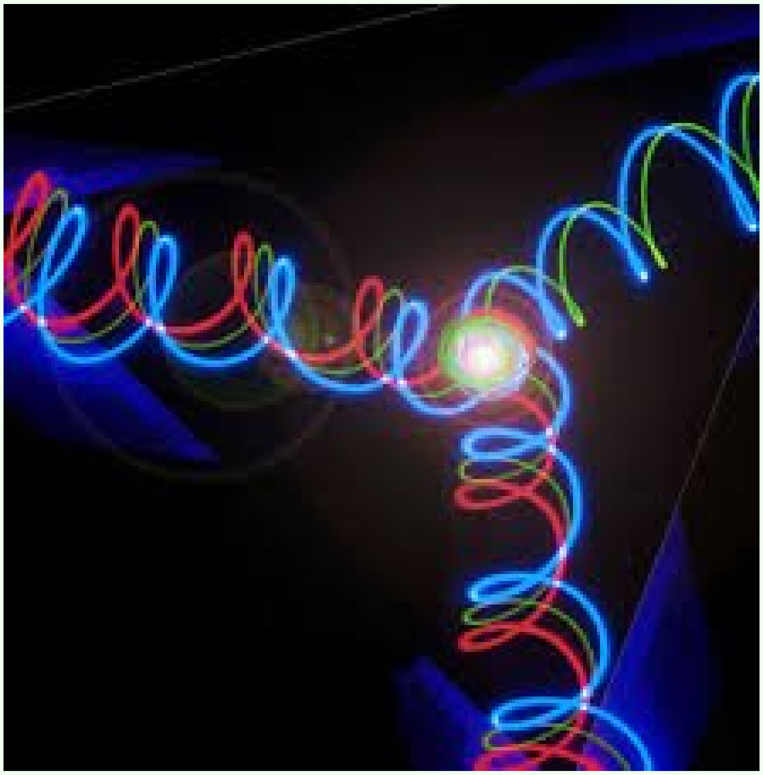


Figure 9: The dynamics of the three photon decay has been worked some time ago.

Briefly, the matrix element $\mathcal{M}$ is

$$\mathcal{M} = -\sqrt{2}V_3$$

for $S_z = 0$ and

$$\mathcal{M} = \pm V_1 + iV_2$$

for $S_z = \pm$. The vector function $\mathbf{V}$ is a lengthy expression. But an important one for the entanglement. $\mathbf{V}$ is given by

$$\begin{aligned} & \mathbf{V}(\mathbf{k}_1, \lambda_1; \mathbf{k}_2, \lambda_2; \mathbf{k}_3, \lambda_3) = \\ & (\lambda_1 - \lambda_2)(\lambda_2 + \lambda_3)\boldsymbol{\epsilon}^*(\hat{\mathbf{k}}_1, \lambda_1)\left[\boldsymbol{\epsilon}^*(\hat{\mathbf{k}}_2, \lambda_2)) \cdot \boldsymbol{\epsilon}^*(\hat{\mathbf{k}}_3, \lambda_3))\right] \\ + \quad & (\lambda_2 - \lambda_3)(\lambda_3 + \lambda_1)\boldsymbol{\epsilon}^*(\hat{\mathbf{k}}_2, \lambda_2)\left[\boldsymbol{\epsilon}^*(\hat{\mathbf{k}}_3, \lambda_3)) \cdot \boldsymbol{\epsilon}^*(\hat{\mathbf{k}}_1, \lambda_1))\right] \\ + \quad & (\lambda_3 - \lambda_1)(\lambda_1 + \lambda_2)\boldsymbol{\epsilon}^*(\hat{\mathbf{k}}_3, \lambda_3)\left[\boldsymbol{\epsilon}^*(\hat{\mathbf{k}}_1, \lambda_1)) \cdot \boldsymbol{\epsilon}^*(\hat{\mathbf{k}}_2, \lambda_2))\right] \end{aligned}$$

This function encodes the whole dynamics including the entanglement. For instance,

$$V(\mathbf{k}_1, \pm; \mathbf{k}_2, \pm; \mathbf{k}_3, \pm) = 0$$

On the other hand

$$\mathbf{V}(\mathbf{k}_1, +; \mathbf{k}_2, +; \mathbf{k}_3, -) = 2\boldsymbol{\epsilon}^*(\hat{\mathbf{k}}_3, -)\left(1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2\right)$$

$$\mathbf{V}(\mathbf{k}_1, +; \mathbf{k}_2, -; \mathbf{k}_3, +) = 2\boldsymbol{\epsilon}^*(\hat{\mathbf{k}}_2, -)\left(1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_3\right)$$

etc. This leads directly to the coefficients of the entanglement.

$$S_z = 0 : \mathcal{M} = -\sqrt{2} V_3$$

and

$$\epsilon_3^*(\hat{\mathbf{k}}_i, \lambda = \pm) = -\sin\theta_i$$

we obtain the unnormalized three parties entangled state

$$\begin{aligned} |\Psi\rangle_{S_z=0} &= \gamma_0 \left[|++-\rangle - |--+\rangle \right] \\ &+ \beta_0 \left[|+-+\rangle - |-+-\rangle \right] \\ &+ \alpha_0 \left[|-++\rangle - |+--\rangle \right] \end{aligned}$$

The coefficients come out as

$$\begin{aligned}
\gamma_0 &= \sin\theta_3 \left[1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2\right] \\
\beta_0 &= \sin\theta_2 \left[1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_3\right] \\
\alpha_0 &= \sin\theta_1 \left[1 - \hat{\mathbf{k}}_2 \cdot \hat{\mathbf{k}}_3\right]
\end{aligned}$$

On the other hand with

$$S_z = \pm : \mathcal{M} = \pm V_1 + iV_2$$

and

$$\epsilon_1^*(\hat{\mathbf{k}}_i, \lambda = \pm) + i\epsilon_2^*(\hat{\mathbf{k}}_i, \lambda = \pm) = e^{i\Phi_i}(\cos\theta_i \pm 1)$$

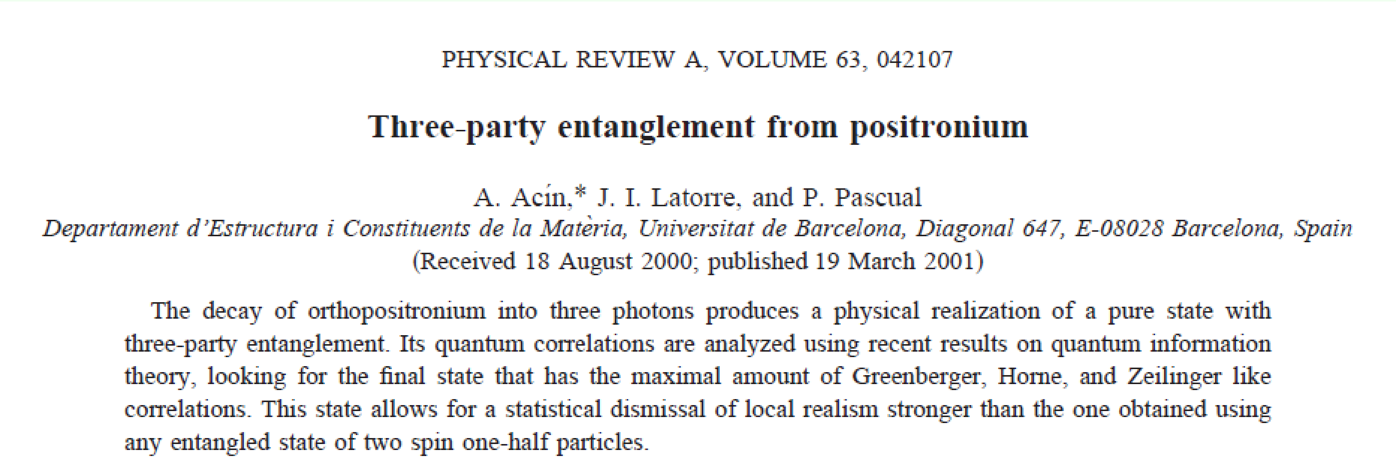
the entangled state is

$$\begin{aligned}
|\Psi\rangle_{S_z=\pm 1} &= \gamma_\pm^{(1)}|++-\rangle + \gamma_\pm^{(2)}|--+\rangle \\
&+ \beta_\pm^{(1)}|+-+\rangle + \beta_\pm^{(2)}|-+-\rangle \\
&+ \alpha_\pm^{(1)}|-++\rangle + \alpha_\pm^{(2)}|+--\rangle
\end{aligned}$$

with

$$
\begin{aligned}
\gamma_{\pm}^{(1)} &= e^{\pm i\Phi_3}\left(\cos\theta_3 - 1\right)\left[1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2\right] \\
\gamma_{\pm}^{(2)} &= e^{\pm i\Phi_3}\left(-\cos\theta_3 - 1\right)\left[1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_2\right] \\
\beta_{\pm}^{(1)} &= e^{\pm i\Phi_2}\left(\cos\theta_2 - 1\right)\left[1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_3\right] \\
\beta_{\pm}^{(2)} &= e^{\pm i\Phi_2}\left(-\cos\theta_2 - 1\right)\left[1 - \hat{\mathbf{k}}_1 \cdot \hat{\mathbf{k}}_3\right] \\
\alpha_{\pm}^{(1)} &= e^{\pm i\Phi_1}\left(\cos\theta_2 - 1\right)\left[1 - \hat{\mathbf{k}}_2 \cdot \hat{\mathbf{k}}_3\right] \\
\alpha_{\pm}^{(2)} &= e^{\pm i\Phi_1}\left(-\cos\theta_2 - 1\right)\left[1 - \hat{\mathbf{k}}_2 \cdot \hat{\mathbf{k}}_3\right]
\end{aligned}
$$

It is only if we choose a coordinate system such that $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ the expression becomes simpler. It makes sense now to introduce you to the paper



PHYSICAL REVIEW A, VOLUME 63, 042107

Three-party entanglement from positronium

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The decay of orthopositronium into three photons produces a physical realization of a pure state with three-party entanglement. Its quantum correlations are analyzed using recent results on quantum information theory, looking for the final state that has the maximal amount of Greenberger, Horne, and Zeilinger like correlations. This state allows for a statistical dismissal of local realism stronger than the one obtained using any entangled state of two spin one-half particles.

Figure 10: This is the first (and still one of the very few) paper on three-photon entanglement from positronium. We will refer to it as ALP

where indeed the authors choose $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, i.e., $\theta_i = \pi/2$.

$$|\psi_0(\hat{k}_1,\hat{k}_2,\hat{k}_3)\rangle = (1-\hat{k}_1\cdot\hat{k}_2)(|++-\rangle + |--+\rangle) + (1-\hat{k}_1\cdot\hat{k}_3)(|+-+\rangle + |-+-\rangle) + (1-\hat{k}_2\cdot\hat{k}_3)(|-++\rangle + |+--\rangle), \quad (19)$$

when the third component of the orthopositronium spin S_z, is equal to zero, and

$$|\psi_1(\hat{k}_1,\hat{k}_2,\hat{k}_3)\rangle = (1-\hat{k}_1\cdot\hat{k}_2)(|++-\rangle - |--+\rangle) + (1-\hat{k}_1\cdot\hat{k}_3)(|+-+\rangle - |-+-\rangle) + (1-\hat{k}_2\cdot\hat{k}_3)(|-++\rangle - |+--\rangle) \quad (20)$$

Figure 11: The three-photon entangled state after putting $\theta_i = 0$

In the case of $S_z = \pm 1$ we do not recover the expression in ALP. We differ by the phases $e^{\pm i\Phi_i}$. Apart form that we have a different assignation for $S_z = 0, \pm 1$. This means that our coefficients depend explicitly on the coordinates (Φ_i) even if we take $\theta_i = \pi/2$.

How important are the phases? If, in addition to choosing the z-axis perpendicular to the three photon plane, we make a rotation of the x-y coordinates, we can get rid of one phase. Factorizing a second phase (which becomes global) we are certainly left with one relative phase. Let me show you how already in a simpler system (two particle entanglement) a phase will play a role in correlation functions which enter the Bell's inequalities. Let me deform a spin- singlet by writing

$$|00\rangle_\alpha = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - e^{i\alpha}|\downarrow\rangle|\uparrow\rangle\right)$$

then

$$\begin{aligned}&{}_\alpha\langle 00|(\hat{\mathbf{a}}\cdot\mathbf{S}^{(1)}\hat{\mathbf{b}}\cdot\mathbf{S}^{(2)})|00\rangle_\alpha = \\ &-\frac{1}{4}\left[\cos\alpha\cos\theta + a_z b_z(1-\cos\alpha) - (\mathbf{a}\times\mathbf{b})_z\sin\alpha\right]\end{aligned}$$

with the internal variable $\cos\theta = \hat{\mathbf{a}}\cdot\hat{\mathbf{b}}$ and (external) variable a_i and b_i (coordinates of the unit vectors).

In correlation functions the phases $e^{\pm i\Phi_i}$ will certainly play a role.

Incidentally, with linear polarized photons reference P. G. Kwiat et al., Phys. Rev. Lett. **75** 4337 (1995) writes a two photon entanglement as

$$\frac{1}{\sqrt{2}}\left[|H\rangle|V\rangle + e^{i\alpha}|V\rangle|H\rangle\right]$$

where $|H\rangle = (1,0)$ and $|V\rangle = (0,1)$ and the matrices used in correlations are Pauli matrices.

A related question comes then into mind. Shall/can we choose $\hat{\mathbf{n}} = \hat{\mathbf{z}}$? For one single event this is certainly possible. If we do this for every subsequent event we will not be able to classify the entanglements according to the spin projections of the positronium $S_z = 0, \pm 1$ since we keep on changing the quantization axis then. If we group together, say $|\Psi\rangle_{S_z=0}$ with $|\Psi\rangle_{S_{z'}=0}$ we might be comparing apples with pears.

Apart from this argument, there is also a loss of generality. Let us take the case $S_z = 0$. The condition that the state factorizes (giving rise to two-particle entanglement) is

$$\beta = 0,\ \gamma = \pm\alpha$$
$$\gamma = 0,\ \beta = \pm\alpha$$
$$\alpha = 0,\ \beta = \pm\alpha$$

For $\theta_i = \pi/2$ choosing one of the three cases above we get a configuration at the edge of the allowed phase space: two collinear momenta and with third one anti-parallel, e.g. for $\gamma = 0$ and $\beta = \alpha$ together with the constraint of the energy-momentum conservation we have

$$\hat{\mathbf{k}}_1{\cdot}\hat{\mathbf{k}}_2 = \cos\theta_{12} = -1,\ \ \hat{\mathbf{k}}_2{\cdot}\hat{\mathbf{k}}_3 = \cos\theta_{23} = \hat{\mathbf{k}}_1{\cdot}\hat{\mathbf{k}}_3 = \cos\theta_{13} = 1$$

Taking now $\theta_i \neq 0$ in general and $\theta_2 = \theta_3 \equiv \theta$ and $\theta_{13} = \theta_{12}$, $\theta_1 = 0$ (which corresponds to the case $\alpha = 0$, $\gamma = \beta$) we get a different configuration which factorizes. This factorizing “star” configuration is (i) not possible if we set all $\theta_i = \pi/2$ right from the beginning and (ii) in accordance with the energy-momentum conservation if

$$k_2 = k_3,\ |cos\theta| = \frac{k_1}{m_e - k_1},\ k_1 \leq \frac{m_e}{2}$$

This could mean that the entanglement properties are different. More specifically, in three-party entanglement there are two non-biseparable classes: W-class and GHZ-class (Greenberger-Horne-Zeilinger) which cannot be transformed into each other by local operations. Generically, one writes

$$\begin{aligned} |GHZ\rangle &= \frac{1}{\sqrt{2}}\left[|000\rangle + |111\rangle\right] \\ |W\rangle &= \frac{1}{3}\left[|001\rangle + |010\rangle + |100\rangle\right] \end{aligned}$$

If we have a three-party entanglement $|\phi\rangle = \sum_{ijk} c_{ijk}|ijk\rangle$ an invariant measure of the entanglement is the so-called hyper-determinant

$$0 \leq Hdet(c_{ijk}) = c_{000}^2 c_{111}^2 + ... + ... c_{000}c_{110}c_{001} + ... \leq 1/4$$

If $Hdet = 0$ and the state does not factorize then we have a W-class entanglement. If we choose $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ the condition $Hdet = 0$ leads to factorization, i.e., the configuration of two collinear and one anti-parallel three momentum which we had before. Since by our choice of the z-axis we lost some generality, could it be that we can reach the W-class in accordance with the energy-momentum conservation? The constraint of the energy-momentum conservation makes the problem more complex than anticipated. First steps: change of basis, form circular to linear polarization $|\pm\rangle = |R/L\rangle$, $|H/V\rangle = |0/1\rangle$

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + i|1\rangle\right]$$

Then

$$\begin{aligned}
|\Psi\rangle_{S_z=0} &= (\alpha_0 + \beta_0 - \gamma_0)\,|010\rangle \\
&+ (\alpha_0 - \beta_0 - \gamma_0)\,|100\rangle \\
&+ (-\alpha_0 - \beta_0 + \gamma_0)\,|001\rangle \\
&+ (\alpha_0 + \beta_0 + \gamma_0)\,|111\rangle
\end{aligned}$$

It is easy to calculate the hyperdetereminant in this basis

$$
\begin{aligned}
Hdet &= (-\alpha_0+\beta_0-\gamma_0)(\alpha_0-\beta_0-\gamma_0) \\
&\cdot\ (-\alpha_0-\beta_0+\gamma_0)(\alpha_0+\beta_0+\gamma_0)
\end{aligned}
$$

If, for instance, $\alpha+\beta+\gamma=0$ the entangled state would look like the W-state even without any change of basis. However, the main question is whether this is possible in accordance with energy-momentum conservation. This is our future task.

This does not exhaust all possibilities how we can write the entangled state. Without much ado and further explanations, but with reference to W. Bernreuther and O. Nachtmann, "Weak interaction in positronium", Z. Phys. **C11** (1981) 235 we define first linear parallel (P) and transversely (T) polarizations with respect to the three-photon plane by

$$\boldsymbol{\epsilon}_{i,P}=\hat{\mathbf{n}}\times\hat{\mathbf{k}}_i,\ \ \boldsymbol{\epsilon}_{i,T}=\hat{\mathbf{n}}$$

Then the entangled state can be written as

$$
\begin{aligned}
|\Psi\rangle &= \frac{16m_e}{k_1k_2k_3}(\hat{\mathbf{n}}\cdot\boldsymbol{\varepsilon})[k_1k_2+k_2k_3+k_1k_3-m_e^2]|TTT\rangle \\
&- \frac{8m_e}{k_1k_2k_3}(\hat{\mathbf{n}}\times\boldsymbol{\varepsilon})[k_1\mathbf{k}_1+k_2\mathbf{k}_2+k_3\mathbf{k}_3]|PPP\rangle \\
&- \frac{8m_e}{k_1k_2k_3}(\hat{\mathbf{n}}\times\boldsymbol{\varepsilon})[k_2\mathbf{k}_1+k_1\mathbf{k}_2]|TTP\rangle \\
&+ \frac{16m_e}{k_1k_2k_3}(\hat{\mathbf{n}}\cdot\varepsilon)[-m_e^2+m_ek_3+k_1k_2]|PPT\rangle
\end{aligned}
$$

where $\boldsymbol{\varepsilon}$ is the polarization vector of the positronium such that if its spin projection S_z with respect to the quantization axis $\hat{\mathbf{z}}$ is zero $\boldsymbol{\varepsilon}=\hat{\mathbf{z}}$ etc.

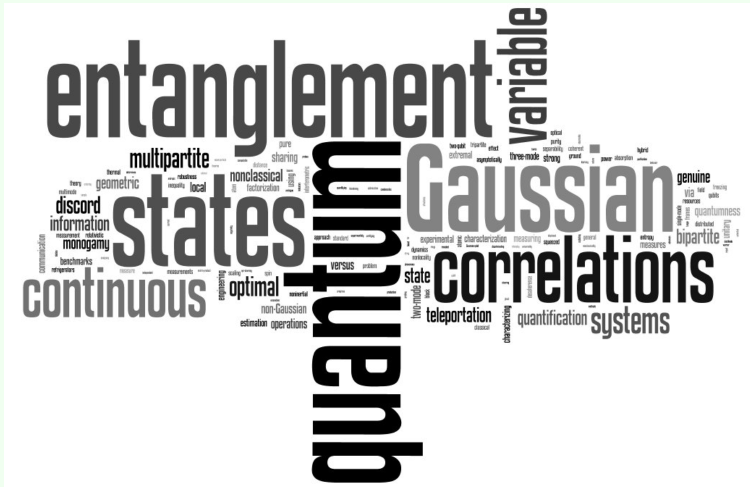

Figure 12: We can test the three-photon entanglement by using inequalities which involve correlations

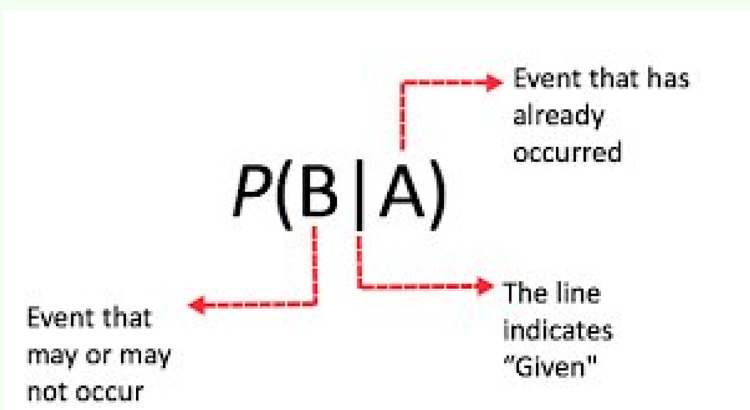

Figure 13: or (conditional) probabilities

Let me come first to the correlations.

Figure 14: It is not really a problem, but a suggestion

What do we mean exactly when we write, say $|++-\rangle$? Obviously, the tensor product

$$|++-\rangle = |+\rangle \otimes |+\rangle \otimes |-\rangle$$

But what exactly is, say $|+\rangle$ or what is its representation in a finite dimensional Hilbert space? Consider the following:

1. We have a two-level system given by the two degrees of freedom of the photon polarization $\lambda = \pm 1$.

2. The photons carry momenta and are entangled in their polarizations, hence

$$|+\rangle = |+\rangle_1 = |\hat{\mathbf{k}}_1, +\rangle = |\boldsymbol{\epsilon}(\hat{\mathbf{k}}_1, +1)\rangle$$

But the polarization vectors are three dimensional objects. The mismatch between the number of qubits $\pm$ and the dimension of the state vector comes from the fact that photons, strictly speaking, do not have spin (defined only in the rest frame). However, one often sees the the correlations functions in the form

$$\langle\Psi|(\hat{\mathbf{a}}\cdot\boldsymbol{\sigma}^{(1)})(\hat{\mathbf{b}}\cdot\boldsymbol{\sigma}^{(2)})(\hat{\mathbf{c}}\cdot\boldsymbol{\sigma}^{(3)})|\Psi\rangle$$

with σ_i the two dimensional Pauli matrices (this correlation enters inequalities like the Mermin or Svetlichny inequality). This implies that the state vectors are also two dimensional. This goes back to the so-called Jones formalism where all polarization states are represented as two dimensional objects. In general

$$|\psi\rangle = (\cos\phi e^{i\alpha_x}, \sin\phi e^{i\alpha_y})$$

If the difference between the phases is $\pi/2$ and the amplitudes equal ($\cos\phi = \sin\phi$) we get the circular polarization states

$$\frac{1}{\sqrt{2}}(1, i), \;\; \frac{1}{\sqrt{2}}(1, -i)$$

This is in accordance with the polarization vectors if $\theta = 0$ (and not $\pi/2$) i.e. we choose $\hat{\mathbf{k}} = \hat{\mathbf{z}}$. Indeed, we have then

$$\begin{aligned} \boldsymbol{\epsilon}(\hat{\mathbf{k}}, +) &= \frac{e^{-i\Phi}}{\sqrt{2}}(1, i, 0) \\ \boldsymbol{\epsilon}(\hat{\mathbf{k}}, -) &= \frac{e^{-i\Phi}}{\sqrt{2}}(1, -i, 0) \end{aligned}$$

which is effectively two dimensional. However, we cannot choose for all photons $\hat{\mathbf{k}} = \hat{\mathbf{z}}$ especially when we have chosen once $\theta_i = \pi/2$ like in ALP. For two photons with $\mathbf{k}_1 + \mathbf{k}_2 = 0$, say in the decay of para-positronium with the entanglement

$$|\Psi\rangle_{para} = \frac{1}{\sqrt{2}}\left[|++\rangle - |--\rangle\right]$$

we can do that if we choose $\hat{\mathbf{k}}_i = \hat{\mathbf{z}}$. With three photons we will have to choose an appropriate operator (corresponding to,say circular polarizations). We can do that by going to the adjoint representation of $SU(2)$

$$(S_i)_{jk} = -i\epsilon_{ijk}$$

The eigenvectors to S_3 are

$$\frac{1}{\sqrt{2}}(1, \pm i, 0)$$

which are the circular polarizations states in a plane perpendicular to the z-axis. A third eigenvector $(0, 0, 1)$ is possible, but the photon will not have it. The eigenvectors to S_1 and S_2 are

$$\frac{1}{\sqrt{2}}(0, \pm i, 1),\ \ \frac{1}{\sqrt{2}}(\pm i, 0, 1)$$

respectively. The first state describes circular polarized photons in a plane perpendicular to the x-axis etc. It makes then sense to consider $\hat{\mathbf{a}} \cdot \mathbf{S}$ and to calculate expectation values of the form

$$\langle \Psi | (\hat{\mathbf{a}} \cdot \mathbf{S}^{(1)})(\hat{\mathbf{b}} \cdot \mathbf{S}^{(2)})(\hat{\mathbf{c}} \cdot \mathbf{S}^{(3)}) | \Psi \rangle$$

Figure 15: Maybe it is just a complicated way to express the “old” correlations.

The Mermin and Svetlichny inequalities, which are a results of local realistic theories for three particles, use correlations. It seems it is also possible to formulate such inequalities using probabilities.

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Extreme Violation of Local Realism in Quantum Hypergraph States

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Hypergraph states form a family of multiparticle quantum states that generalizes the well-known concept of Greenberger-Horne-Zeilinger states, cluster states, and more broadly graph states. We study the nonlocal properties of quantum hypergraph states. We demonstrate that the correlations in hypergraph states can be used to derive various types of nonlocality proofs, including Hardy-type arguments and Bell inequalities for genuine multiparticle nonlocality. Moreover, we show that hypergraph states allow for an exponentially increasing violation of local realism which is robust against loss of particles. Our results suggest that certain classes of hypergraph states are novel resources for quantum metrology and measurement-based quantum computation.

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Remark 2.—Putting together all the null terms derived from the stabilizer formalism and subtracting the terms causing a Hardy-type argument, we obtain the Bell inequality

$$\begin{aligned}\langle \mathcal{B}_3^{(1)} \rangle = {} & [P(+--|XZZ) + P(-++|XZZ) \\ & + P(-+-|XZZ) + P(--+|XZZ) + \text{permutations}] \\ & - [P(+--|XXX) + \text{permutations}] \geq 0, \end{aligned} \tag{10}$$

where the permutations include all distinct terms that are obtained by permuting the qubits. The three-uniform hypergraph state violates the inequality (11) with the value of $\langle \mathcal{B}_3^{(1)} \rangle = -3/16$.

This Bell inequality follows from the Hardy argument: If a deterministic local model predicts one of the results with the minus signs, it also has to predict at least one of the results corresponding to the terms with a plus sign, otherwise, it contradicts with the Hardy argument. In addition, all the terms with a minus sign are exclusive, so a deterministic LHV model can predict only one of them.

The Hardy-type argument and the Bell inequality can be generalized to a higher number of qubits, if we consider N-qubit hypergraphs with the single hyperedge having a cardinality N:

This kind of model, even if different bipartitions are mixed, cannot explain the correlations of the hypergraph state, meaning that the hypergraph state is genuine multiparticle nonlocal. First, one can see by direct inspection that the stabilizer conditions from Eqs. (5) and (6) are not compatible with the hypergraph correlations $P(---|XXX) = 1/16$ and $P(---|ZZZ) = 1/8$. Contrary to the correlations in Eq. (9) these are symmetric, and allow the construction of a Bell-Svetlichny inequality [18] valid for all the different bipartitions.

Observation 4.—Putting all the terms from the hypergraph stabilizer formalism and the correlations $P(---|XXX)$ and $P(---|ZZZ)$ together, we obtain the following Bell-Svetlichny inequality for genuine multiparticle nonlocality,

$$\begin{aligned}\langle \mathcal{B}_3^{(2)} \rangle = {} & [P(+--|XZZ) + P(-++|XZZ) \\ & + P(-+-|XZZ) + P(--+|XZZ) + \text{permutations}] \\ & + P(---|XXX) - P(---|ZZZ) \geq 0, \end{aligned} \tag{12}$$

which is violated by the state $|H_3\rangle$ with $\langle \mathcal{B}_3^{(2)} \rangle = -1\backslash 16$.

The proof is done by an exhaustive assignments of nonsignaling and local models.

Figure 16: The work is still in the process...

1. Our three photon entangled states include some phases which the seminal paper of Acin, Lattorre and Pascual do not have.

2. Apart from that we do not put the z-axis perpendicular to the photon plane. This, as shown, gives a bigger freedom and we could probe into the two different classes, W and GHZ class in accordance with the energy-momentum constraint.

3.(Iconoclastic) For three photons we suggested a three dimensional correlation formalism since photons as far as their polarizations are concerned are two level systems with a three dimensional formalism.

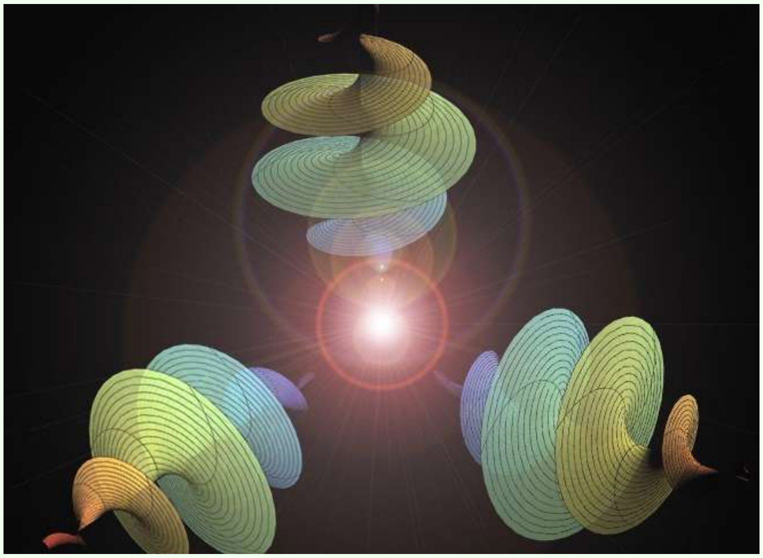

Figure 17: THANK YOU!