Entanglement distribution in random states of identical particles

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Workshop on Discrete Symmetries and Entanglement

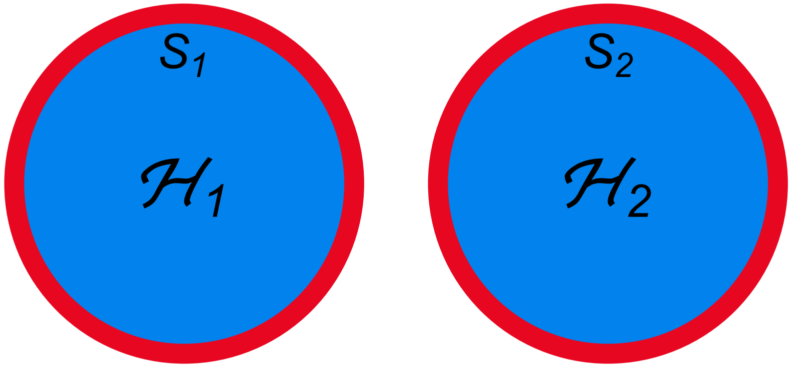
Cracow 10-11.6.2017

entanglement - basics

two qubits

Hilbert space

$$\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{C}^2$$



entanglement - basics

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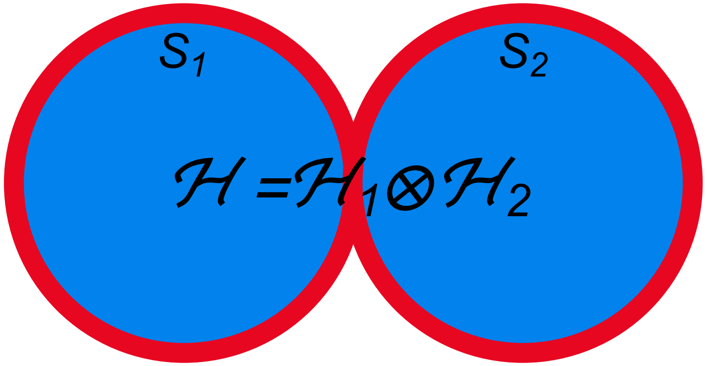

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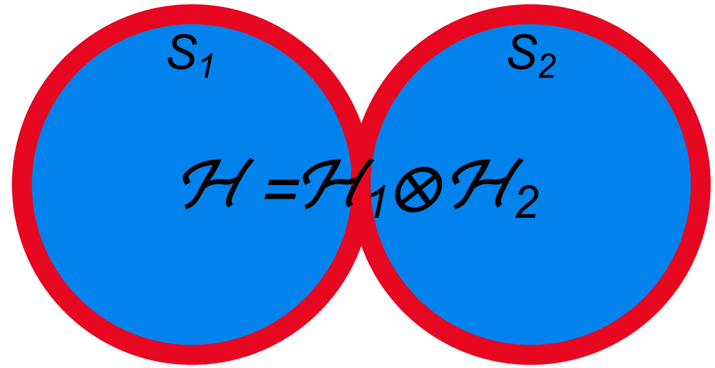
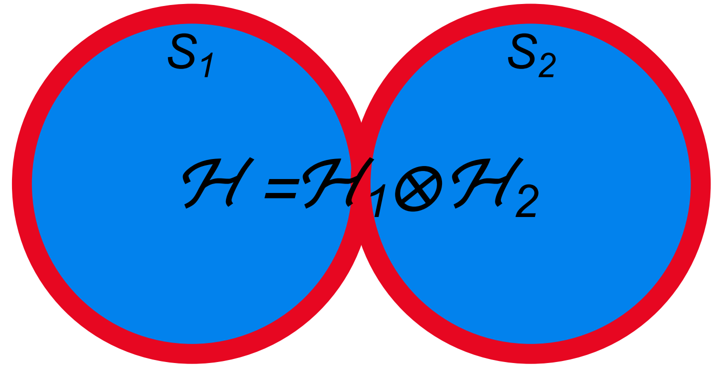


Local operators

$$\sigma_a^{(1)} = a \cdot \sigma \otimes 1 \qquad \sigma_b^{(2)} = 1 \otimes b \cdot \sigma$$

entanglement - basics

two qubits



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$$|\psi\rangle = |\phi\rangle_1 |\varphi\rangle_2$$

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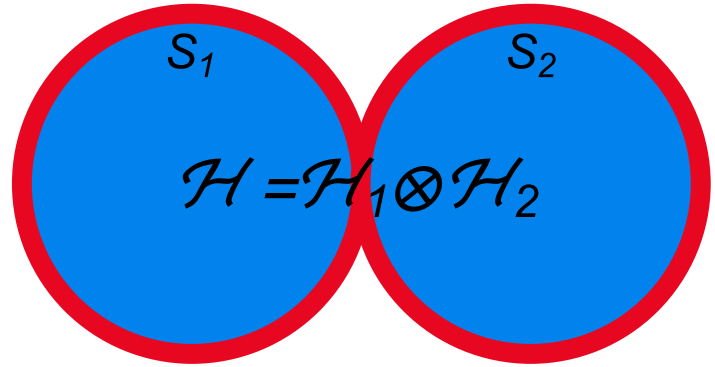

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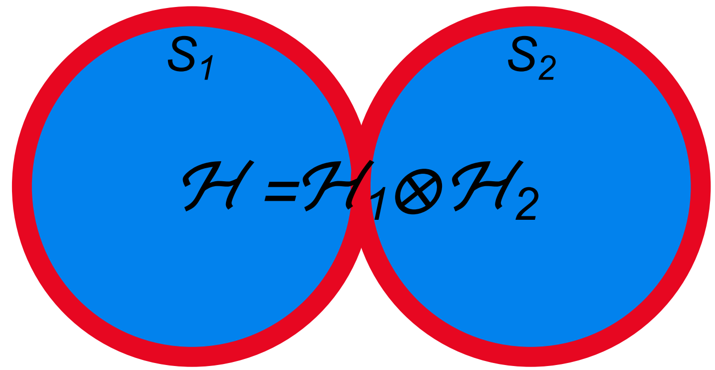
entanglement - basics



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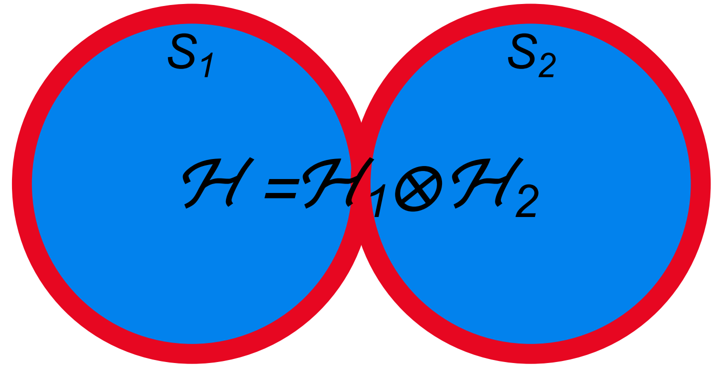
$$|\psi\rangle \neq |\phi\rangle_1 |\varphi\rangle_2$$

entanglement - basics

two qubits



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Entangled state

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Expectation values

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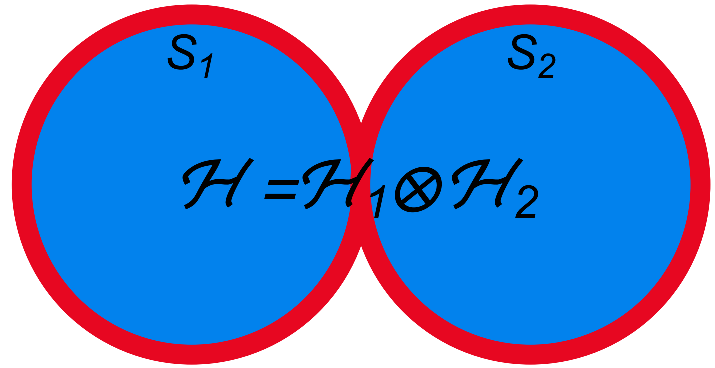

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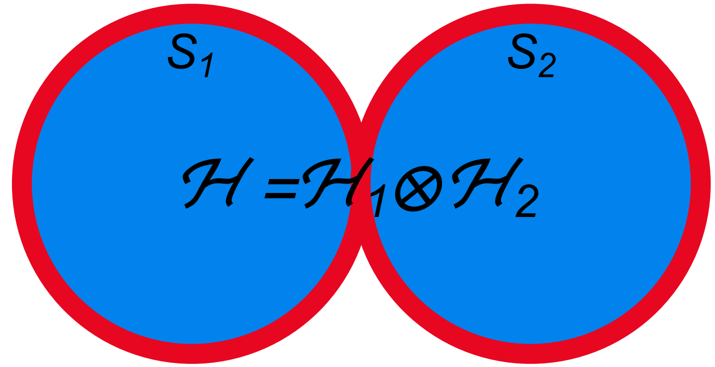
two qubits



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$$\boxed{\text{tensor product structure of Hilbert space} \longrightarrow \text{entanglement}}$$

separability criteria

Schmidt decomposition

$$|\psi\rangle = \sum_{ij} c_{ij} |\phi_i\rangle |\varphi_j\rangle$$

separability criteria

Schmidt decomposition

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singular value
decomposition

separability criteria

Schmidt decomposition

$$|\psi\rangle = \sum_{ij} c_{ij} |\phi_i\rangle |\varphi_j\rangle \xrightarrow[\text{(change of basis)}]{\text{Schmidt decomposition}} |\psi\rangle = \sum_i \lambda_i |\tilde{\phi}_i\rangle |\tilde{\varphi}_i\rangle$$

singular value decomposition

Schmidt coefficients

separability criteria

Schmidt decomposition: if there is more than one Schmidt coefficient different from 0 the state is entangled!

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singular value decomposition

Schmidt coefficients

Entanglement quantification

Entropy

$$E(\psi) = -\mathrm{Tr}[\rho_r \ln \rho_r] = -\sum_i \lambda_i \ln \lambda_i$$

Concurrence

$$C(\psi) = |\langle\tilde{\psi}|\psi\rangle| = \sqrt{2\sum_{i\neq j}\lambda_i\lambda_j}$$

$$|\tilde{\psi}\rangle = D|\psi\rangle = \sigma_y \otimes \sigma_y|\psi^*\rangle$$

time reversal operator

identical particles

- indistinguishability of identical particles
- symmetrization postulate

fermions

anti-symmetric states
(half-integer spin)

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle_1|\beta\rangle_2 - |\beta\rangle_1|\alpha\rangle_2)$$

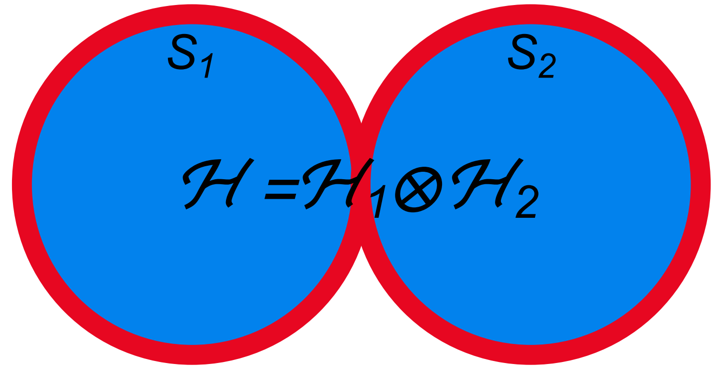
bosons

symmetric states
(integer spin)

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle_1|\beta\rangle_2 + |\beta\rangle_1|\alpha\rangle_2)$$

are these states entangled?
which is the nature of entanglement in identical particle systems?

identical particles



$$\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{C}^n$$

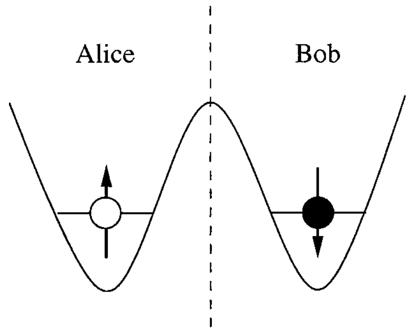
dimension of the composite Hilbert space

distinguishable particles	fermions	bosons
$\mathcal{H} = \mathcal{C}^n \otimes \mathcal{C}^n$	$\mathcal{H} = A[\mathcal{C}^n \otimes \mathcal{C}^n]$	$\mathcal{H} = S[\mathcal{C}^n \otimes \mathcal{C}^n]$
$\mathrm{Dim}(\mathcal{H}) = n^2$	$\mathrm{Dim}(\mathcal{H}) = \dfrac{n(n-1)}{2}$	$\mathrm{Dim}(\mathcal{H}) = \dfrac{n(n+1)}{2}$

symmetrization postulate $\longrightarrow$ No (obvious) tensor product structure for the composite system
¿entanglement?

entanglement as a resource

distinguishable particles



$$| \psi_{\text{init}}\rangle_{AB} = |\phi\rangle|\uparrow\rangle_A \otimes |\chi\rangle|\downarrow\rangle_B$$

[K.Eckert, et al. Annals of Physics 299, 88 (2002)]

entanglement as a resource

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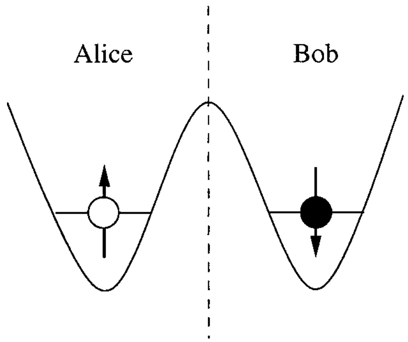
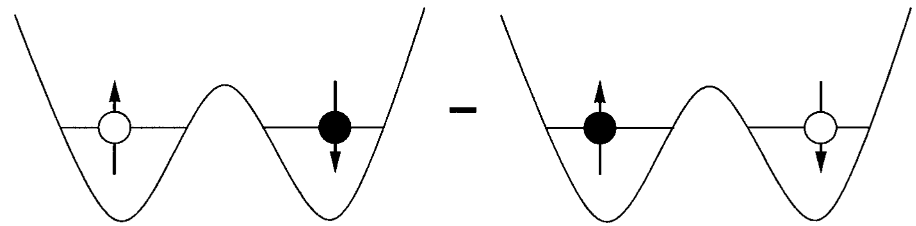


$$| \, \psi_{\text{init}}\rangle_{AB} = |\phi\rangle| \uparrow\rangle_A \otimes |\chi\rangle| \downarrow\rangle_B$$

lower the barrier

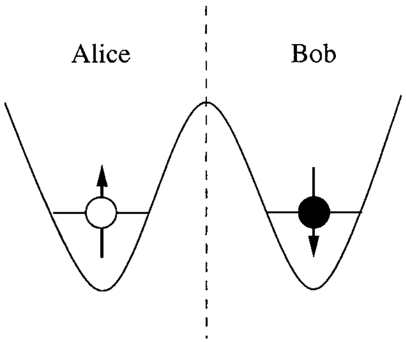
indistinguishable particles



$$| \, \psi(t_1)\rangle_{AB} = \frac{1}{\sqrt{2}}[|\phi\rangle| \uparrow\rangle_1 \otimes |\chi\rangle| \downarrow\rangle_2 - |\chi\rangle| \downarrow\rangle_1 \otimes |\phi\rangle| \uparrow\rangle_2]$$

entanglement as a resource

distinguishable particles



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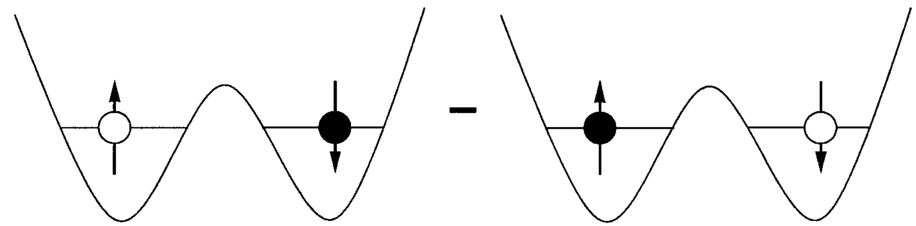

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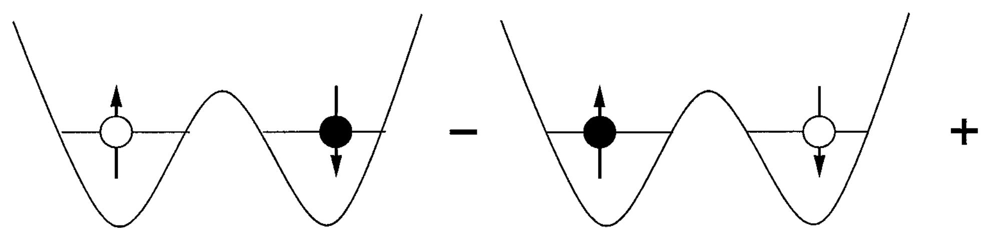
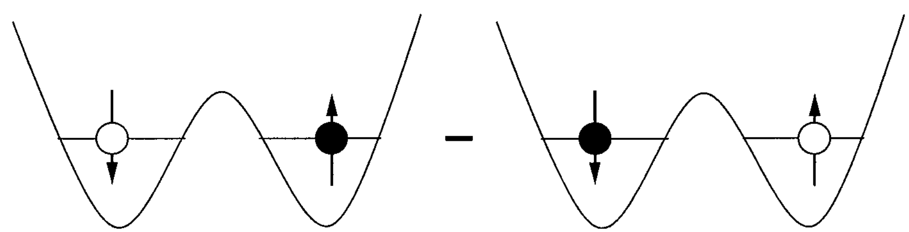
change potential

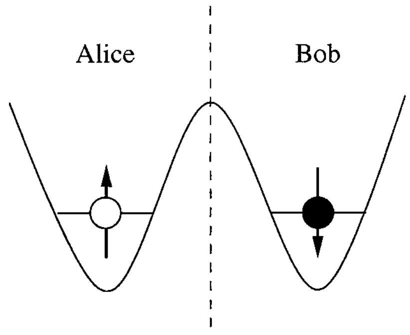




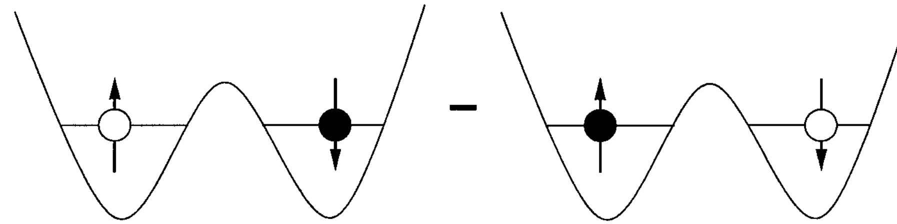
$$|\,\psi(t_2)\rangle_{AB} = \frac{1}{2}[|\phi\rangle|\uparrow\rangle_1 \otimes |\chi\rangle|\downarrow\rangle_2 - |\chi\rangle|\downarrow\rangle_1 \otimes |\phi\rangle|\uparrow\rangle_2$$
$$+|\phi\rangle|\downarrow\rangle_1 \otimes |\chi\rangle|\uparrow\rangle_2 - |\chi\rangle|\uparrow\rangle_1 \otimes |\phi\rangle|\downarrow\rangle_2]$$

distinguishable particles

indistinguishable particles



lower the barrier



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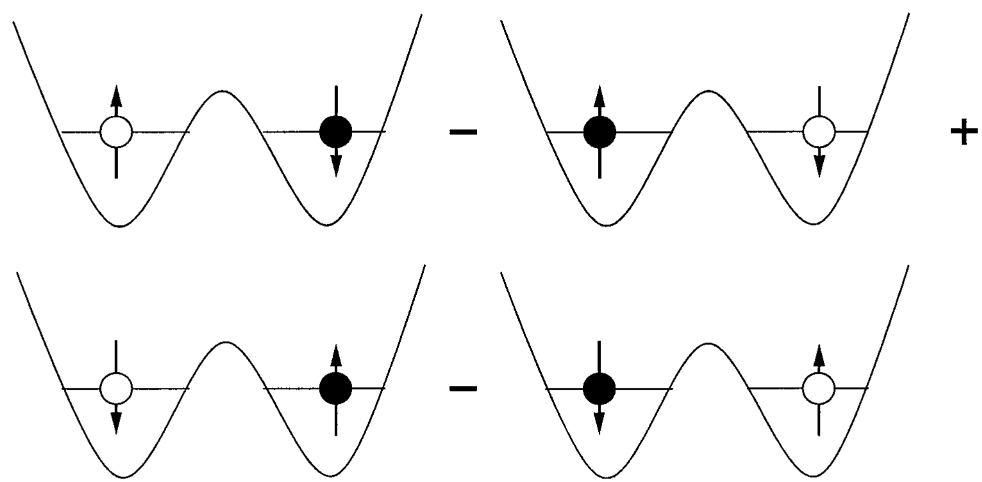
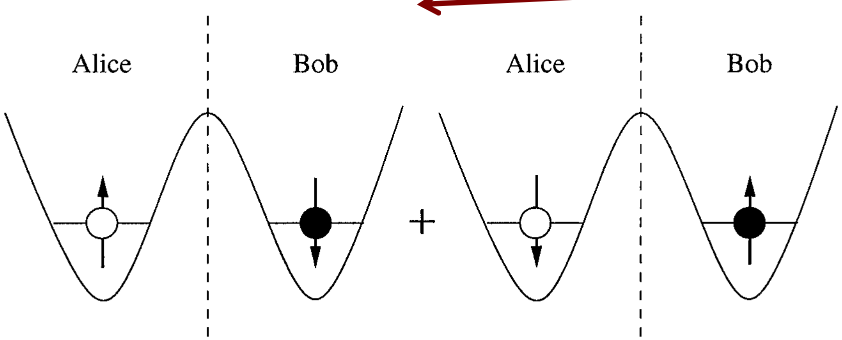
change potential

increase the barrier



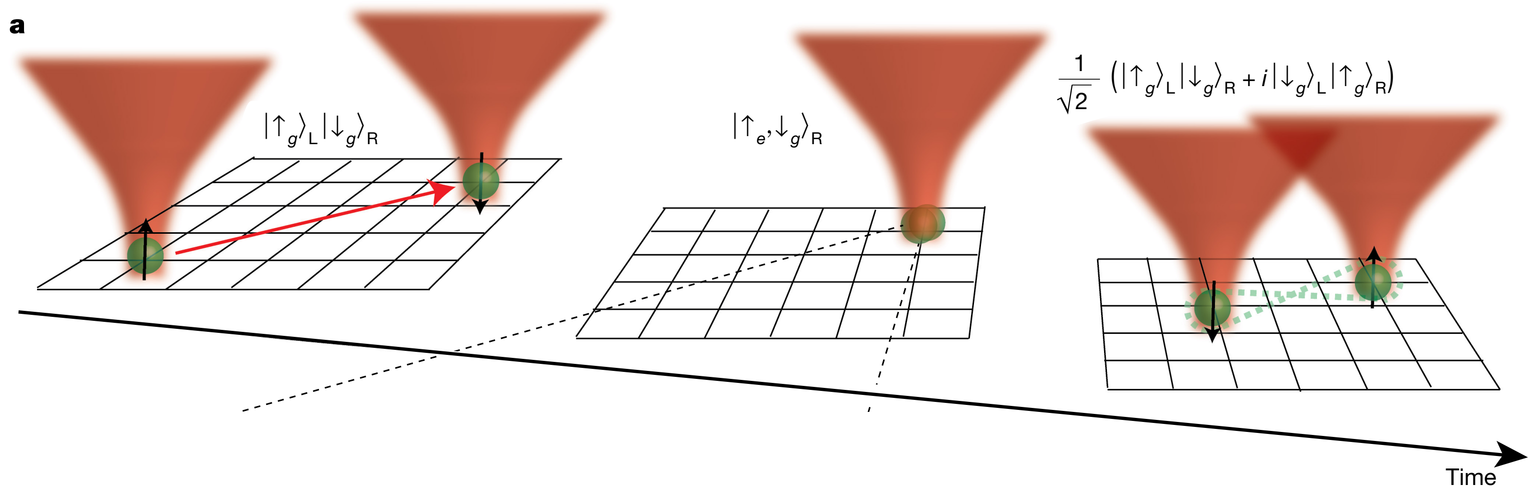


$$|\,\psi_{\text{final}}\rangle_{AB} = \frac{1}{\sqrt{2}}[|\phi\rangle|\uparrow\rangle_A \otimes |\chi\rangle|\downarrow\rangle_B + |\phi\rangle|\downarrow\rangle_A \otimes |\chi\rangle|\uparrow\rangle_B]$$

$$|\,\psi(t_2)\rangle_{AB} = \frac{1}{2}[|\phi\rangle|\uparrow\rangle_1 \otimes |\chi\rangle|\downarrow\rangle_2 - |\chi\rangle|\downarrow\rangle_1 \otimes |\phi\rangle|\uparrow\rangle_2 + |\phi\rangle|\downarrow\rangle_1 \otimes |\chi\rangle|\uparrow\rangle_2 - |\chi\rangle|\uparrow\rangle_1 \otimes |\phi\rangle|\downarrow\rangle_2]$$

Entangling via local spin exchange

Neutral atoms in optical tweezers



[A.M. Kaufman et al, Nature 527, 208–211 (2015)]

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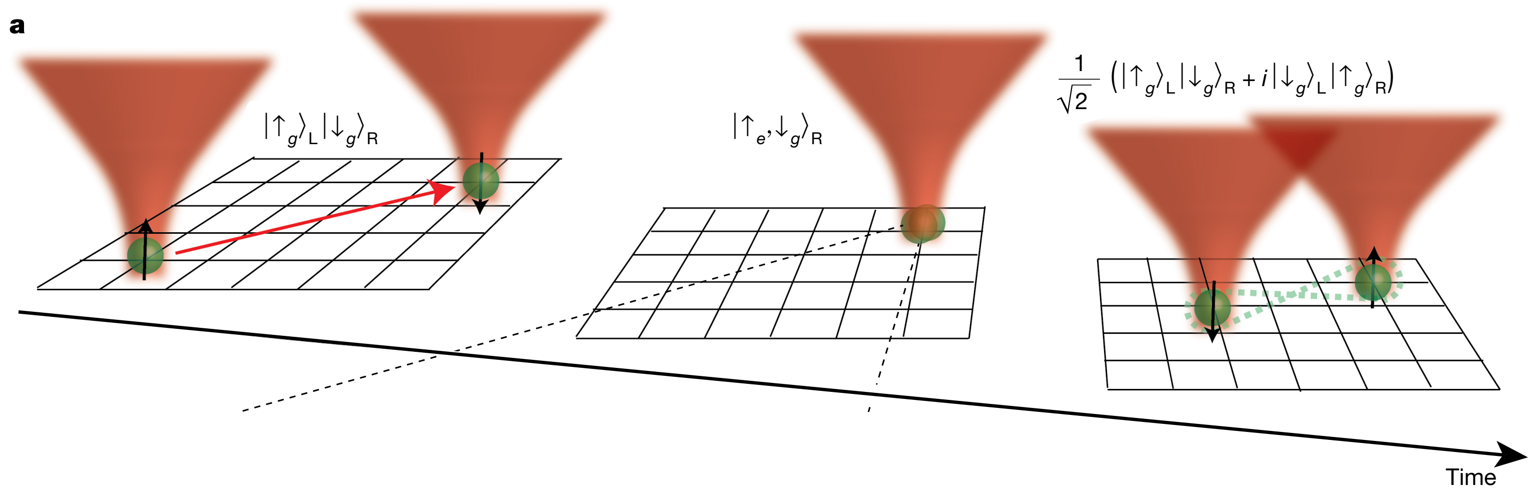

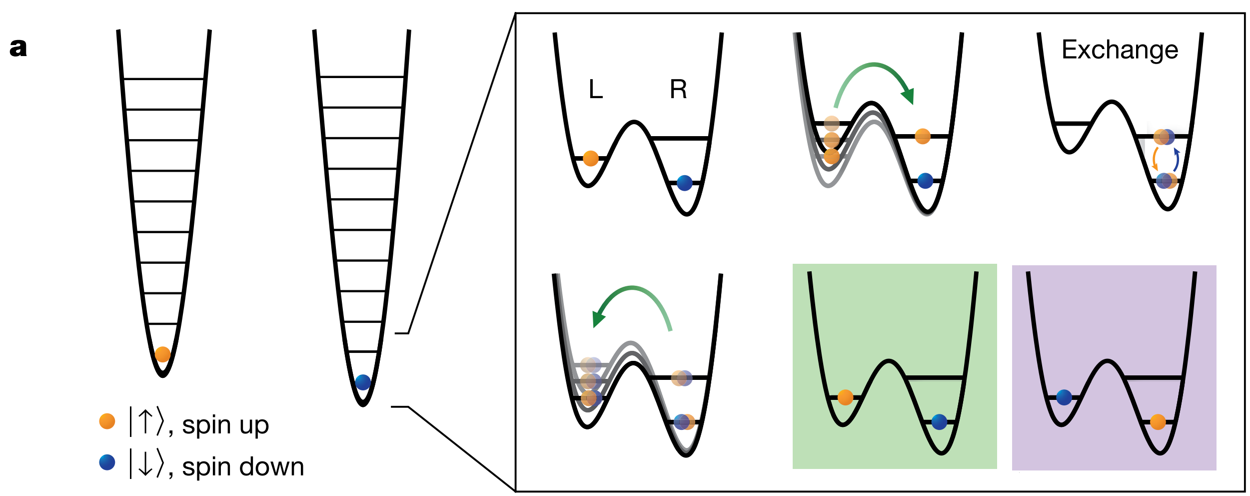

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indistinguishable particles: entanglement

Fermions

$$|\psi\rangle = \sum_{i,j}^{2K} w_{i,j} f_i^\dagger f_j^\dagger |0\rangle$$

[John Schliemann, et al. PRA 64, 022303 (2001)]
[R.Paskauskas, L.You. PRA 64, 042310 (2001)]

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block diagonal

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Schmidt-Slater coefficients

product states

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Is this formal analogy enough?

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diagonal

Is this formal analogy enough? YES!

[N. Killoran, et al. PRL 112, 150501 (2014)]
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entanglement quantification

concurrence

(lowest dimensional systems)

- distinguishable particles

  two-level

  $\mathcal{H}_{\rm S} = \mathcal{C}^2$ $\quad$ $\mathcal{H} = \mathcal{H}_{\rm S} \otimes \mathcal{H}_{\rm S}$, $\mathrm{Dim}(\mathcal{H}) = 4$ $\quad$ $C(|\psi\rangle) = 2|\psi_{12}\psi_{21} - \psi_{22}\psi_{11}|$

- fermions

  four-level

  $\mathcal{H}_{\rm S} = \mathcal{C}^4$ $\quad$ $\mathcal{H} = A(\mathcal{H}_{\rm S} \otimes \mathcal{H}_{\rm S})$, $\mathrm{Dim}(\mathcal{H}) = 6$ $\quad$ $C(|w\rangle) = 8|w_{12}w_{34} + w_{13}w_{24} + w_{14}w_{23}|$

- bosons

  two-level

  $\mathcal{H}_{\rm S} = \mathcal{C}^2$ $\quad$ $\mathcal{H} = S(\mathcal{H}_{\rm S} \otimes \mathcal{H}_{\rm S})$, $\mathrm{Dim}(\mathcal{H}) = 3$ $\quad$ $C(|\psi\rangle) = 4|v_{11}v_{22} - v_{12}^2|$

[K.Eckert, et al. Annals of Physics 299, 88 (2002)]

entanglement quantification

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(lowest dimensional systems)

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$$C(|\psi\rangle) = 2|\psi_{12}\psi_{21} - \psi_{22}\psi_{11}| = 2|\det(\psi)|$$

- fermions

  four-level

$$\mathcal{H}_{\mathrm{S}} = \mathcal{C}^4 \qquad \mathcal{H} = A(\mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{S}}) \qquad \mathrm{Dim}(\mathcal{H}) = 6$$

$$C(|w\rangle) = 8|w_{12}w_{34} + w_{13}w_{24} + w_{14}w_{23}| = 8|\det(w)|^{1/2}$$

- bosons

  two-level

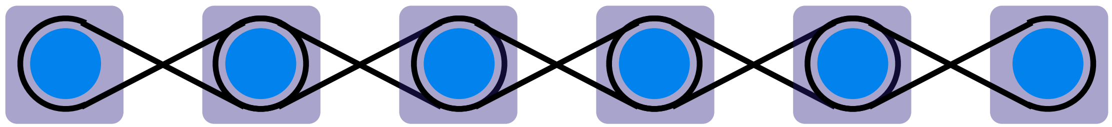
$$\mathcal{H}_{\mathrm{S}} = \mathcal{C}^2 \qquad \mathcal{H} = S(\mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{S}}) \qquad \mathrm{Dim}(\mathcal{H}) = 3$$

$$C(|\psi\rangle) = 4|v_{11}v_{22} - v_{12}^2| = 4|\det(v)|$$

[K.Eckert, et al. Annals of Physics 299, 88 (2002)]

entanglement quantification

distinguishable particles



SL-invariant measure

special linear group $G \equiv \mathrm{SL}(d_1, \mathbb{C}) \otimes \mathrm{SL}(d_2, \mathbb{C}) \otimes \cdots \otimes \mathrm{SL}(d_N, \mathbb{C})$

the group $\mathrm{SL}(d, \mathbb{C})$ of $d \times d$ matrices with determinant one.

- invariance $\mathcal{E}_{\mathrm{inv}}(\hat{g}\rho\hat{g}^\dagger) = \mathcal{E}_{\mathrm{inv}}(\rho)$ for $g \in G$
- homogeneity $\mathcal{E}_{\mathrm{inv}}(r\rho) = r\mathcal{E}_{\mathrm{inv}}(\rho)$ for $r > 0$
- mixed states $\mathcal{E}_{\mathrm{inv}}(\rho) = \min \sum_i p_i \mathcal{E}_{\mathrm{inv}}(\psi_i)$

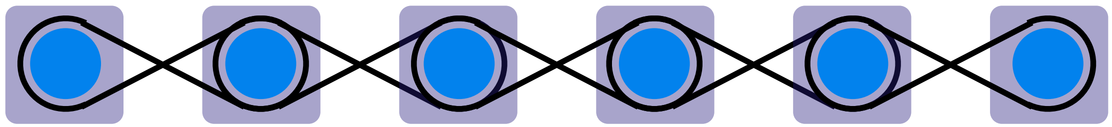
examples

- concurrence
- three tangle (three-qubit)
- G-Concurrence (bipartite systems)

[G. Gour, PRA 71, 012318 (2005)]

entanglement quantification

distinguishable particles



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examples

- concurrence
- three tangle (three-qubit)
- G-Concurrence (bipartite systems)

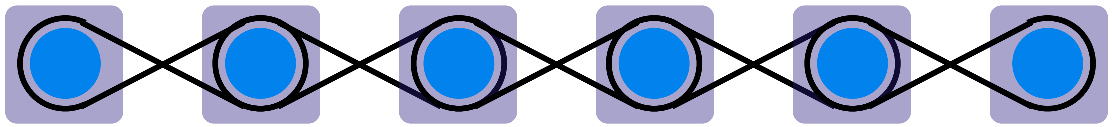
$$|\psi\rangle = \sum_{ij} \psi_{ij} |\phi_i\rangle |\varphi_j\rangle$$

$$\boxed{G_d(|\psi\rangle) = d|\det(\psi^\dagger\psi)|^{1/d}}$$

[G. Gour, PRA 71, 012318 (2005)]

entanglement quantification

identical particles



SL-invariant measure

special linear group: single particle space $G \equiv \mathrm{SL}(d_1, \mathbb{C})$

the group $\mathrm{SL}(d, \mathbb{C})$ of $d \times d$ matrices with determinant one.

- invariance $\mathcal{E}_{\mathrm{inv}}(\hat{g}\rho\hat{g}^\dagger) = \mathcal{E}_{\mathrm{inv}}(\rho)$ for $g \in G$
- homogeneity $\mathcal{E}_{\mathrm{inv}}(r\rho) = r\mathcal{E}_{\mathrm{inv}}(\rho)$ for $r > 0$
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examples

- G-Concurrence (bipartite systems)

entanglement quantification

G-Concurrence

(higher dimensional systems)

fermions

$$|w\rangle = \sum_{i,j}^{2K} w_{i,j} f_i^\dagger f_j^\dagger |0\rangle \longrightarrow |\psi_A\rangle = \sum_{i,j}^{2K} \psi_{ij} |ij\rangle \qquad \psi_{ij} = -\psi_{ji} = \sqrt{2} w_{ij}$$

$$\boxed{G_{d=2K}(|\psi_A\rangle) = d|\det(\psi_A)|^{2/d} = 2d|\det(w)|^{2/d}}$$

bosons

$$|v\rangle = \sum_{i,j}^{K} v_{i,j} b_i^\dagger b_j^\dagger |0\rangle \longrightarrow |\psi_s\rangle = \sum_{i,j}^{K} \psi_{ij} |ij\rangle \qquad \psi_{ij} = \psi_{ji} = \sqrt{2} v_{ij}$$

$$\boxed{G_{d=K}(|\psi_S\rangle) = d|\det(\psi_S)|^{2/d} = 2d|\det(v)|^{2/d}}$$

bipartite random states

random states of fermionic and bosonic systems

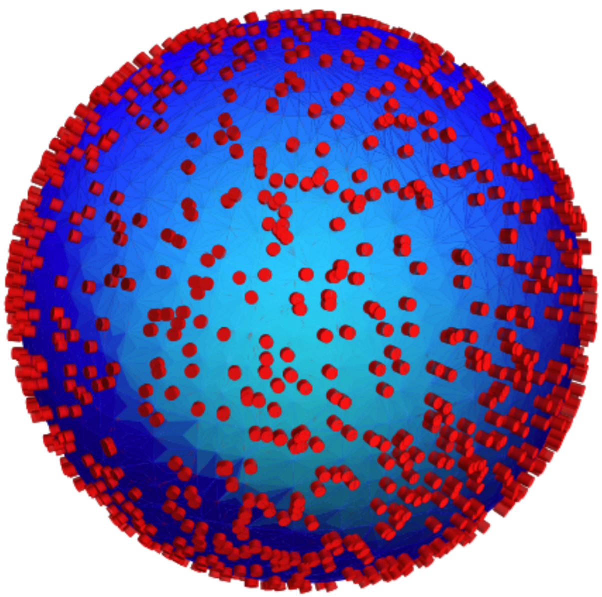
why random states?

- entanglement is an useful resource for quantum computation and randomness is a way to create it.
- useful in super dense coding, remote state preparation, data hiding protocols.
- they provide a natural benchmarks.
- random states allow to asset general behaviors with minimal prior information.

states distribution: uniform

normalized pure states uniformly distributed on the Hilbert space (Haar measure)

$$U_{d_T} \in \mathrm{CUE}$$



$$P(\Psi) = P(\psi_1, \ldots, \psi_2) = \mathcal{N}_d \; \delta(1 - |\Psi|^2)$$

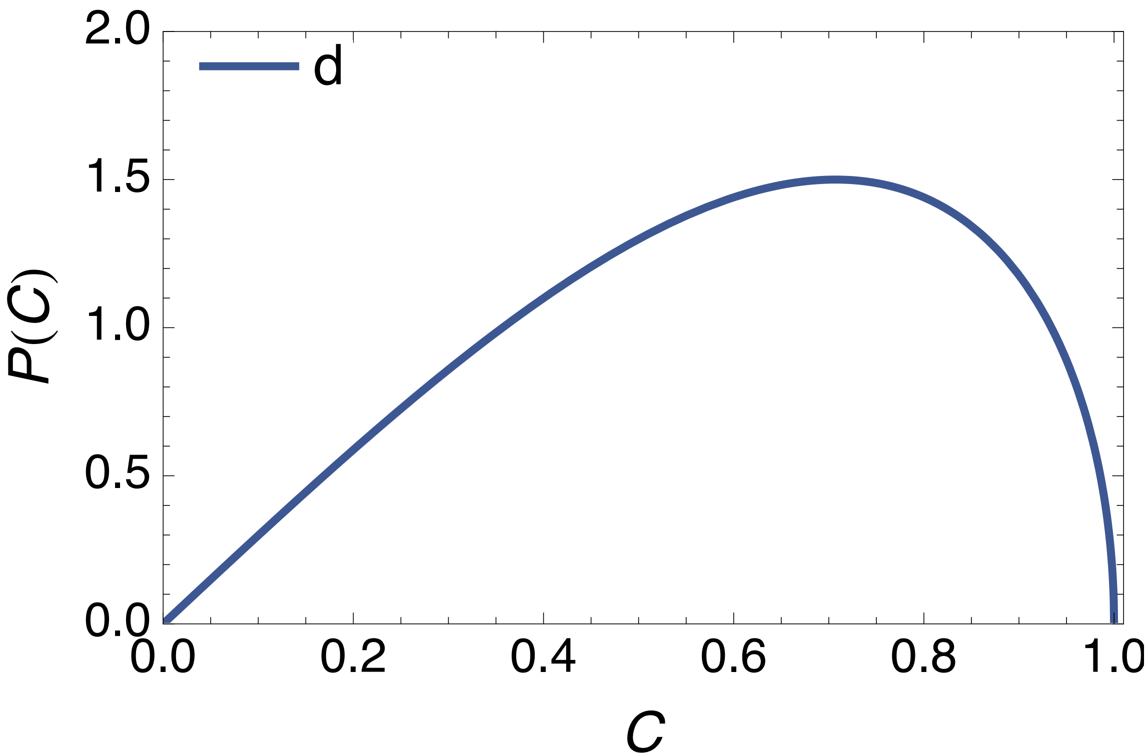
G-concurrence distribution

$$P(G_d) = \int [d\psi] \delta \left(G_d - d |\det(\psi\psi^\dagger)|^{1/d} \right) P(\psi)$$

G-concurrence distribution: simplest case

distinguishable particles

$$P(C) = 3C(1 - C^2)^{1/2}$$



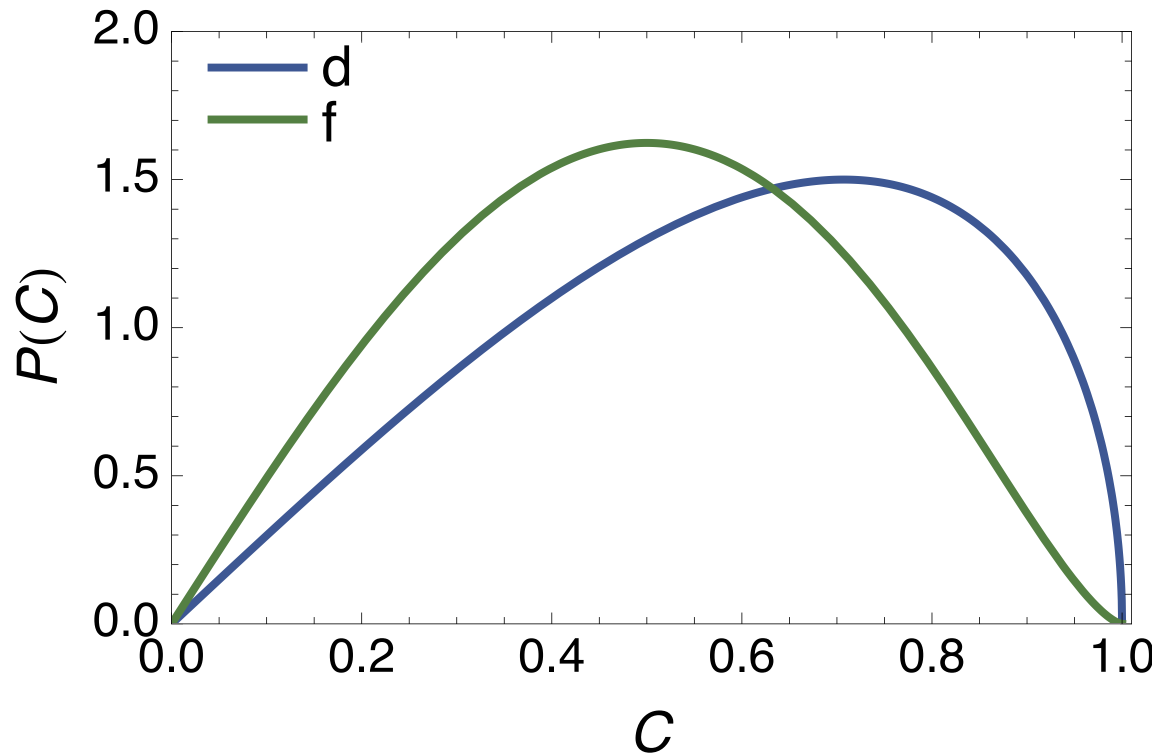
G-concurrence distribution: simplest case

distinguishable particles

$$P(C) = 3C(1-C^2)^{1/2}$$

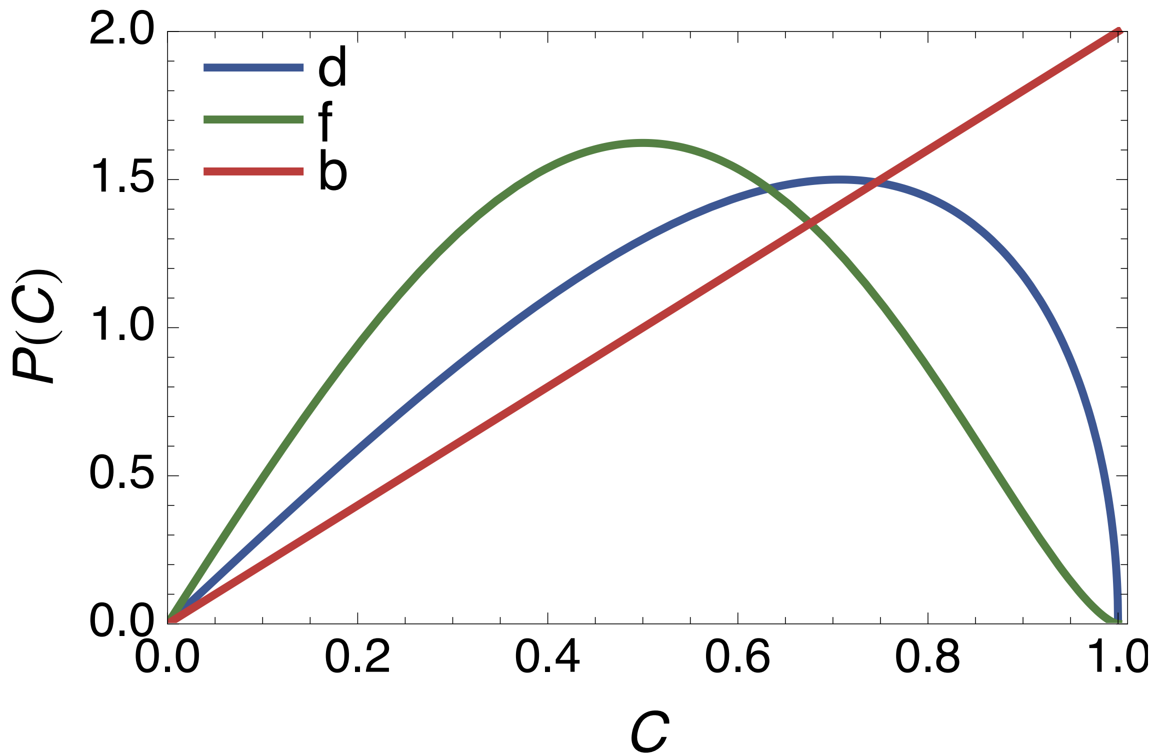
fermions

$$P_f(C) = 5C(1-C^2)^{3/2}$$



G-concurrence distribution: simplest case

distinguishable particles	fermions	bosons
$P(C) = 3C(1-C^2)^{1/2}$	$P_f(C) = 5C(1-C^2)^{3/2}$	$P_b(C) = 2C$



G-concurrence distribution

$$\boxed{G_d(|\psi\rangle) = d|\det(\psi^\dagger\psi)|^{1/d} = d|\prod_i \lambda_i|^{2/d}}$$

$$P(G_d) = \int [d\psi]\delta\left(G_d - d|\det(\psi\psi^\dagger)|^{1/d}\right) P(\psi)$$

G-concurrence distribution

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Join probability density of Slater-Schmidt coefficients

$$P_N^{(c)}(\lambda_1,\ldots,\lambda_N) := \mathcal{C}_N^{(c)}\delta\left(1-\sum_i \lambda_i\right)\prod_{i=1}^{N}\theta(\lambda_i)\prod_{i<j}|\lambda_i-\lambda_j|^{2\gamma}$$

$\gamma(c) = 2^c$ and $c = 0$ distinguishable, $c = -1$ bosons, and $c = +1$ fermions.

[K. Zyczkowski and H.-J. Sommers, J. Phys. a: Math. Gen. 34, 7111 (2001)]

[V. Cappellini et al, PRA 74(6), 062322 (2006)]

G-concurrence distribution

Caos de Colombia

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Moments of $G_{(c)N}$

$$\langle G_{(c)}^M\rangle_N = N^M\langle D_{(c)}^{M/N}\rangle_N = N^M\frac{\Gamma\left(N+\gamma N(N-1)\right)}{\Gamma\left(N+M+\gamma N(N-1)\right)}\prod_{j=0}^{N-1}\frac{\Gamma(1+M/N+\gamma j)}{\Gamma(1+\gamma j)}$$

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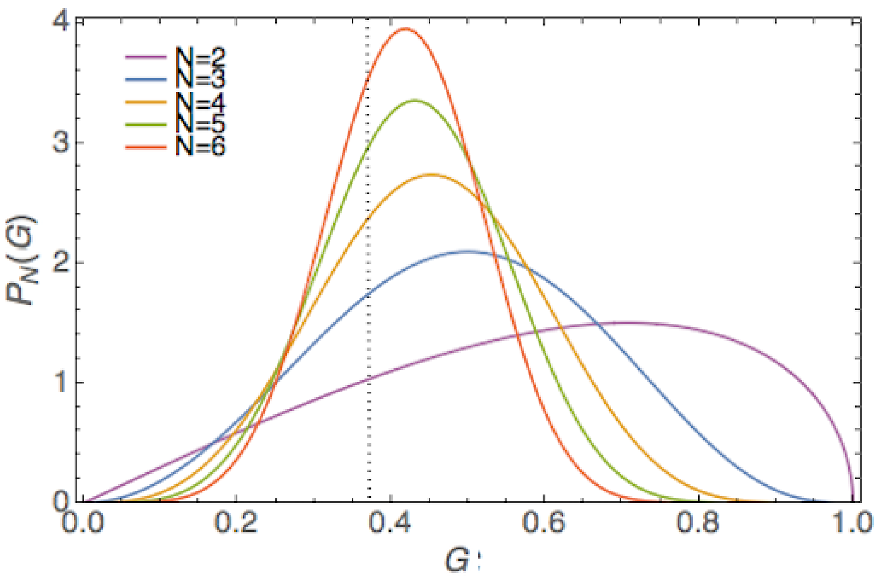
G-concurrence distribution

distinguishable particles



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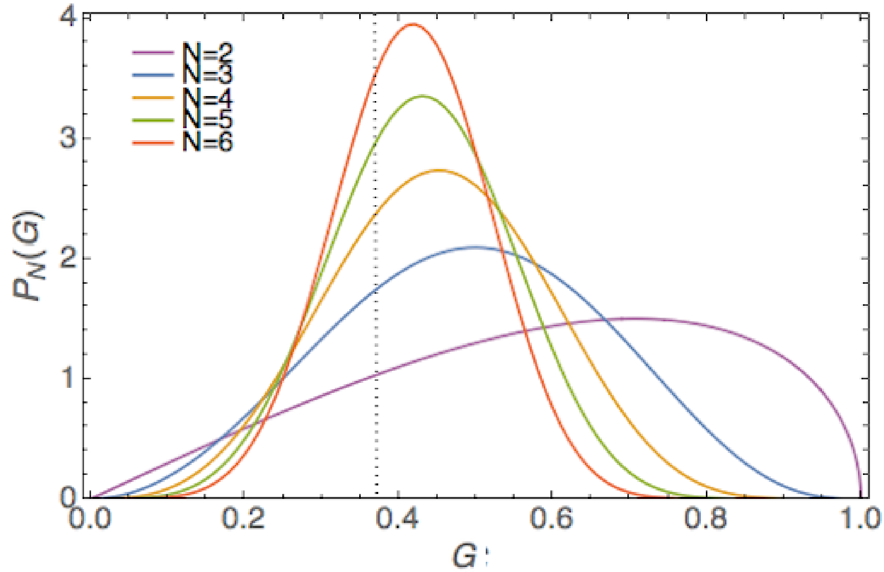
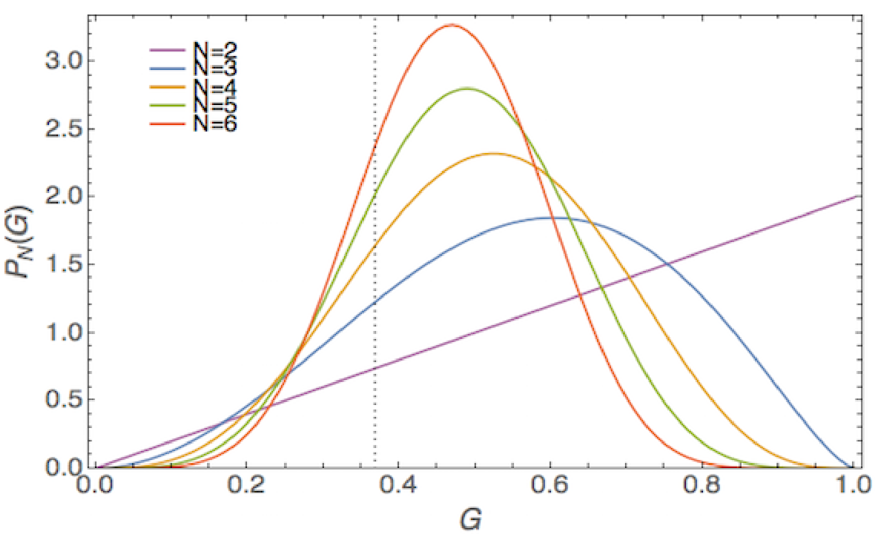
G-concurrence distribution

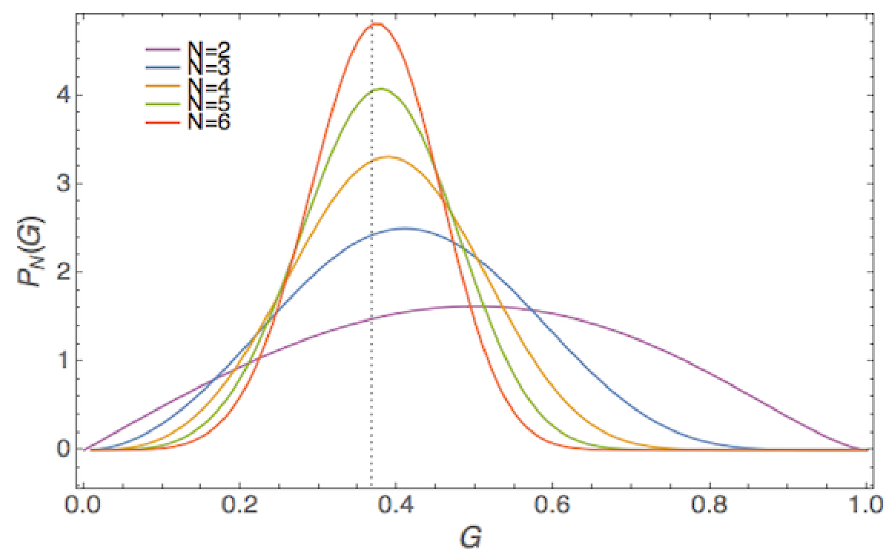
distinguishable particles



bosons



fermions



outlook

conclusions

- we identified the restriction of the SL-invariant measures to the symmetric and anti-symmetric subspaces as possible measures for entanglement in systems of indistinguishable particles.
- we used G-concurrence to study the distribution of entanglement in bipartite systems of indistinguishable particles.

outlook

- extension of our ideas to multipartite systems (three tangle)
- extension of our ideas to mixed states